Theorem on altitudes and the Jacobi identity

A. Zaslavskiy and M. Skopenkov

Solutions.

First let us give a table containing the answers to all the problems:

Algebraic object	Geometric sense
$\vec{A}$	a point A
$\vec{a}$	a spherical line a
$[\vec{A},\vec{B}]$	the line passing through A and B
$[\vec{a},\vec{b}]$	the intersection point of a and b
$[\vec{A},\vec{b}]$	the perpendicular dropped from the point A to the line b
$\vec{A}+\vec{B}$	the middle of the segment AB
$(\vec{a},\vec{b})$	cosine of the angle between a and b
$\lvert[\vec{a},\vec{b}]\rvert$	sine of the angle between a and b
$(\vec{A},\vec{B})$	cosine of the length of the segment AB (i. e. the arc of a big circle between A and B)
$\lvert[\vec{A},\vec{B}]\rvert$	sine of the length of the segment AB
$(\vec{A},\vec{b})$	sine of the distance between A and b
$\lvert[\vec{A},\vec{b}]\rvert$	cosine of the distance between A and b
$\vec{A}+\vec{B}+\vec{C}=0$	1) the points A, B and C are collinear 2) the centers of the circles A, B and C are collinear
$\vec{a}+\vec{b}+\vec{c}=0$	the lines a, b and c have a common point
$[\vec{A},[\vec{B},\vec{C}]]+[\vec{B},[\vec{C},\vec{A}]]+[\vec{C},[\vec{A},\vec{B}]]=0$	the altitudes of a triangle ABC have a common point
$[\vec{A},\vec{B}+\vec{C}]+[\vec{B},\vec{C}+\vec{A}]+[\vec{C},\vec{A}+\vec{B}]=0$	the medians of a triangle ABC have a common point
$[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]$	the center of the circumscribed circle of the triangle ABC
$(\vec{A},\vec{B},\vec{C})$	sine of the length of the segment AB $\cdot$ sine of the distance between C and AB
$(\vec{A},[\vec{B},\vec{C}])=(\vec{B},[\vec{C},\vec{A}])=(\vec{C},[\vec{A},\vec{B}])$	The sines theorem for the triangle ABC
$(\vec{A},\vec{B})=1$	the circles A and B are orthogonal
$\frac{[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]}{(\vec{A},\vec{B},\vec{C})}$	the circle orthogonal to the circles A, B and C
$\vec{A}-\vec{B}+\vec{C}-\vec{D}=0$	there exists a circle orthogonal to the circles A, B, C and D
$\frac{d_B}{d_B-d_A}\vec{A}-\frac{d_A}{d_B-d_A}\vec{B}$	bisector of the circles A and B

8. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be the vectors corresponding to the vertices of the spherical triangle ABC. By Problem 1 the vector $[\vec{A},\vec{B}]$ corresponds to the spherical line passing through the points A and B. Then by Problem 3 the vector $[\vec{A},[\vec{B},\vec{C}]$ corresponds to the perpendicular dropped from the point A to the line BC, that is, it corresponds to the line containing the altitude h_A of the triangle ABC. Analogously the vectors $[\vec{B},[\vec{C},\vec{A}]$ and $[\vec{C},[\vec{A},\vec{B}]$ correspond to the lines containing two other altitudes of the triangle ABC. So by Problem 5 the Jacobi identity means that the three constructed lines have a common point, i. e. we get the theorem on altitudes of a spherical triangle.

9. Denote by $\vec{a}$ *the direction vector* of the line a, that is, a vector parallel to the line a. Analogously, let $\vec{b}$ and $\vec{c}$ be the direction vectors of the lines b and c. Then obviously the vector $[\vec{b},\vec{c}]$ is parallel to the common perpendicular to the lines b and c, that is, to the line a'. Thus the vector $[\vec{a},[\vec{b},\vec{c}]]$ is parallel to the common perpendicular to the lines a and a', that is, to the line a''. Analogously, the vectors $[\vec{b},[\vec{c},\vec{a}]]$ and $[\vec{c},[\vec{a},\vec{b}]]$ are parallel to the lines b'' and c'' respectively. By the Jacobi identity the last three vectors are parallel to one plane. Thus the three lines a'', b'' ïS c'' are parallel to one plane.

10. By symmetry the vector $\vec{A}+\vec{B}$ corresponds to the middle of the arc of the big circle passing through the points A and B. It is natural to say that this point is *the middle* of the spherical segment with ends A and B.

Remark. In fact a pair of points A and B on the sphere determines two distinct segments. If *one* of the two vectors $\vec{A}$ and $\vec{B}$ is replaced by the opposite vector, then their sum corresponds to the middle of the other segment with ends A and B.

11. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be unit vectors pointing to A, B and C respectively from the center of the sphere. Then by Problem 10 the vector $\vec{B}+\vec{C}$ corresponds to the middle of the segment BC. Then by Problem 1 the vector $[\vec{A},\vec{B}+\vec{C}]$ corresponds to *the median* of the triangle ABC. Then by Problem 5 the identity $[\vec{A},\vec{B}+\vec{C}]+[\vec{B},\vec{C}+\vec{A}]+[\vec{C},\vec{A}+\vec{B}]=0$ means that the medians of the triangle ABC have a common point.

The theorem on medians can also be proved by consideration of the vector $\vec{A}+\vec{B}+\vec{C}$.

Remark. (A. Mafusalov) Unlike Euclidean geometry, the theorem on medians of a spherical triangle has an external analog: two 'external' medians m_A, m_B and the opposite 'internal' median m_C of a triangle have a common point.

12. First let us prove that the vectors $\vec{A}-\vec{B}$, $\vec{B}-\vec{C}$ and $\vec{C}-\vec{A}$ correspond to the *middle perpendiculars* to the sides of a spherical triangle ABC. Indeed, since

$$(\vec{A}-\vec{B},\vec{A}+\vec{B})=(\vec{A},\vec{A})-(\vec{B},\vec{B})=1-1=0,$$

it follows that the line corresponding to $\vec{A}-\vec{B}$ passes through the middle of AB. And since

$$(\vec{A}-\vec{B},[\vec{A},\vec{B}])=(\vec{A},[\vec{A},\vec{B}])-(\vec{B},[\vec{A},\vec{B}])=0,$$

it follows that this line is orthogonal to the segment AB. Thus the vector $\vec{A}-\vec{B}$ corresponds to the middle perpendicular to the segment AB.
Denote by $\vec{V}=[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]$. Let V be the point on the sphere corresponding to this vector. Let us check that the point V belongs to all the three constructed middle perpendiculars. Indeed, since

$$(\vec{V},\vec{A}-\vec{B})=([\vec{B},\vec{C}],\vec{A})-([\vec{C},\vec{A}],\vec{B})=0,$$

it follows that V belongs to the line corresponding to the vector $\vec{A}-\vec{B}$. Analogously, V belongs to two other middle perpendiculars.
Remark. The identity $(\vec{A}-\vec{B})+(\vec{B}-\vec{C})+(\vec{C}-\vec{A})=0$ and Problem 5 also imply that the three middle perpendiculars to the sides of a triangle have a common point.
13. The solution follows directly from the formulas $(\vec{a},\vec{b})=|\vec{a}|\cdot|\vec{b}|\cdot\cos\gamma$ and $|[\vec{a},\vec{b}]|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\gamma$, where γ is the angle between the vectors $\vec{a}$ and $\vec{b}$.
14. Assume that the lengths of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are equal to 1. Then by Problem 13 we obtain $|([\vec{A},\vec{B}],\vec{C})|=\sin AB\sin h_C$, where h_C is the length of the altitude of the triangle ABC passing through the vertex C. The identity $(\vec{A},[\vec{B},\vec{C}])=(\vec{B},[\vec{C},\vec{A}])=(\vec{C},[\vec{A},\vec{B}])$ means that $\sin AB\sin h_C=\sin BC\sin h_A=\sin CA\sin h_B$ (1). Writing an analogous identity for the unit vectors corresponding to the lines containing the sides of the triangle ABC, we get $\sin\angle C\sin h_C=\sin\angle A\sin h_A=\sin\angle B\sin h_B$ (2). Dividing (1) by (2), we get spherical *sine theorem*:

$$\frac{\sin AB}{\sin\angle C}=\frac{\sin BC}{\sin\angle A}=\frac{\sin CA}{\sin\angle B}.$$

Remark. Mixed product of three vectors has a clear meaning — it is the volume of the parallelepiped spanned by these three vectors, with a sign. We do not know such a natural interpretation of this number in terms of spherical geometry.
15. *First solution.* It is proved analogously to the case Euclidean geometry that *the sides* of a spherical triangle uniquely define the angles of the triangle. Now it remains to consider the triangle dual to the initial one (that is, a triangle such that the vectors pointing to its vertices are orthogonal to the planes containing the sides of the initial triangle).
*Second solution.*Analogously to the solution of Problem 6 one can prove the identity $([\vec{a},\vec{c}],[\vec{b},\vec{c}])=(\vec{a},\vec{b})(\vec{c},\vec{c})-(\vec{a},\vec{c})(\vec{b},\vec{c})$. Applying the results of Problem 13, we obtain the formula (*the spherical cosine theorem*)

$$\sin\angle A\sin\angle B\cos AB=\cos\angle C-\cos\angle A\cos\angle B.$$

This implies directly that the angles of a spherical triangle uniquely determine the sides of the triangle.
Remark.* The obtained results remain true also in the Lobachevskiy (hyperbolic) geometry, because it can be transformed to spherical by multiplication of all coordinates by i. Then the words 'the lines have a common point' in all the formulations above should be replaced by 'the lines belong to one sheaf'. Formally, our above proofs remain valid in this case, if one takes the unit sphere in the quasi-euclidean space and puts $[\vec{A},\vec{B}]:=*(\vec{A}\wedge\vec{B})$ [1].

19. It is easy to check that the directions of both vectors u and v do not depend on the choice of points A and B: the vector u is parallel to our plane, and the vector v is orthogonal to the plane, passing through our line and the point O. The length of the vector u is equal to AB, and the length of the vector v is equal to two areas of the triangle ABO, that is, to $AB\cdot h$, where h is the distance between O and our line. Thus for another choice of the points A and B the vectors u and v are multiplied by the same number.
20. Indeed, our line should lie in the plane passing through the point O and orthogonal to the vector v. Our line should also be parallel to the vector v, and the distance between our line and the point O is equal to $|v|/|u|$. The vector $[u,v]$ points, in which 'direction' with respect to the point O our line lies. Thus our line is uniquely defined by the pair of vectors u and v.
21. The construction of the desired line, starting from the vectors $\vec{u}$ and $\vec{v}$, is in fact given in the solution of the previous problem.
22. (This is one of the most hard problems of this set) Define *the inner product* of bivectors $\hat{a}=(u,v)$ and $\hat{b}=(u',v')$ by the formula

$$(\hat{a},\hat{b})=((u,v);(u,v')+(u',v)).$$

The inner product of two bivectors is a *pair* of numbers. It is easy to check that $(\hat{a},[\hat{a},\hat{b}])=(\hat{b},[\hat{a},\hat{b}])=(0;0)$. So our problem is a direct corollary of the following lemma:

Lemma. If $(\hat{a},\hat{b})=(0;0)$, then the lines a and b intersect, and the angle between them is right.

Proof of the lemma. The lines a and b are orthogonal because the first component of the inner product of $\hat{a}$ and $\hat{b}$ is zero: $(u,u')=0$. It remains to shows that if $(\hat{a},\hat{b})=(0;0)$, then the lines a and b intersect.
First consider the case when $u\perp v$ and $u'\perp v'$.
Let us check that the inner product does not depend on the choice of the point O. Assume that in the beginning we fix another point O_1. Let $\hat{a}_1=(u_1,v_1)$, $\hat{b}_1=(u_1',u_1')$ be the bivectors obtained from the lines a and b by our construction, if we assume that the fixed point is O_1. Then one can check directly that $u_1=u$, $v_1=v+[u,\overrightarrow{OO_1}]$ пїЅ $u_1'=u'$, $v_1'=v'+[u',\overrightarrow{OO_1}]$. So

$$(\hat{a}_1,\hat{b}_1)=((u,u');(u,v'+[u',\overrightarrow{OO_1}])+(v+[u,\overrightarrow{OO_1}],u'))=((u,u');(u,v')+(v,u'))=(\hat{a},\hat{b}),$$

because $(u, [u', \overrightarrow{OO_1}]) + ([u, \overrightarrow{OO_1}], u') = 0$. Thus indeed the inner product of bivectors does not depend on the choice of the point O.
Therefore we may assume without loss of generality that the point O belongs to the line a. Then $v = 0$. Thus the condition $(\hat{a}, \hat{b}) = (0; 0)$ means that the vector u is orthogonal to the vector v'. If $v' = 0$, then we are done, because the point O is a common point for a and b in this case. If $v' \neq 0$, then both lines lie in the plane passing through the point O orthogonal to the vector v', and hence these two lines intersect each other. Thus the case $u \perp v$, $u' \perp v'$ has been considered.
Now consider the case when not necessarily $u \perp v$ and $u' \perp v'$. (This case is important for what follows, because even if this condition is satisfied for the bivectors $\hat{a}$ and $\hat{b}$, it may not be satisfied for the bivector $[\hat{a}, \hat{b}]$.)
In this case, by definition, the bivector $\hat{a} = (u, v)$ corresponds to the same line as the bivector $(u, pr\, v)$. Denote the last bivector by $\hat{a}_\perp$. Notice that $pr\, v = v + \alpha u$ for some number α. Define analogously the bivector $b_\perp$ and the number β. Since $(\hat{a}, \hat{b}) = (0; 0)$, it follows that $(\hat{a}_\perp, \hat{b}_\perp) = ((u, u'); (u, v') + \beta(u, u') + (v, u') + \alpha(u, u')) = (\hat{a}, \hat{b}) = (0; 0)$, because $(u, u') = 0$. Then the second case of the lemma is reduced to the first one, and we are done.
*Remark**. The definition of the product of bivectors comes from the formula for the commutator in the Lie algebra of the movements group of the 3-space. *Sliding vector* is another geometric interpretation of our bivectors [3].
23. Notice that $(\hat{a}+\hat{b}, [\hat{a}, \hat{b}]) = (\hat{a}, [\hat{a}, \hat{b}]) + (\hat{b}, [\hat{a}, \hat{b}]) = 0$. Then by lemma from the solution of Problem 22 the line corresponding to the vector $\hat{a} + \hat{b}$ intersects the line corresponding to the vector $[\hat{a}, \hat{b}]$. And by Problem 22 the last line is the common perpendicular to the lines a and b.
24. The Jacobi identity for the product of bivectors follows directly from the Jacobi identity for the product of vectors.
25. Applying Problem 22 several times, we obtain that the bivectors $[\hat{a}, [\hat{b}, \hat{c}]]$, $[\hat{b}, [\hat{c}, \hat{a}]]$ and $[\hat{c}, [\hat{b}, \hat{a}]]$ correspond to the lines a'', b'' and c'' respectively. By the Jacobi identity it follows that $[\hat{c}, [\hat{b}, \hat{a}]] = [\hat{a}, [\hat{b}, \hat{c}]] + [\hat{b}, [\hat{c}, \hat{a}]]$. Then by Problem 23 the line c'' intersects the common perpendicular to the lines a'' and b''. Denote this common perpendicular by h. By Problem 9 the angle between c'' and h is right. So h is a common perpendicular for a'', b'' and c'', and the Theorem on altitudes of a triangle follows.

27. In this solution we will use some facts which are proved in the further solutions.
It is sufficient to prove the statement of the problem for circles on a sphere (since a plane can be mapped into a sphere by a stereographic projection). For some circle, we will denote the corresponding vector by the same letter with the arrow over it (e.g. the vector corresponding to a circle a is denoted by $\vec{a}$). By problem 34, $\vec{c}' = \frac{d_B}{d_B - d_A}\vec{a} - \frac{d_A}{d_B - d_A}\vec{b}$; one can write down the analogous formulae for $\vec{a}'$ and $\vec{b}'$. Then we have

$$\frac{d_C - d_B}{d_C d_B}\vec{a}' + \frac{d_A - d_C}{d_A d_C}\vec{b}' + \frac{d_B - d_A}{d_B d_A}\vec{c}' = 0.$$

Applying the lemma from the solution of problem 34, we obtain that the circles a, b, and c have a common point.
*Note**. Three circles in this statement can form a "convex" or an "concave" triangle. According to this, our theorem is equivalent to the theorem on bisectors in spherical geometry or in Lobachevskiy geometry.
28*. (Solution by A. Mafusalov) We will present a plan of the solution, omitting some technical details. It is based on the following elegant theorem.

Theorem on the altitudes of a curved triangle. Let A, B, C be pairwise intersecting circles. Let C' be a circle perpendicular to C and passing through two points of intersection of A and B. The circles A' and B' are defined analogously. Then three circles A', B' and C' belong to one sheaf. (In particular, if two of them intersect, then the third passes through their common point(s)).

One can check that the statement of problem 28 is essentially the theorem on altitudes for the curved triangle $a'b'c'$.

Proof of the theorem. Let D be the radical center of A, B, and C. Denote by d the degree of the point D with respect to our circles. Consider the sphere of radius $\sqrt{|d|}/2$ touching the initial plane in D. Let us perform the stereographic projection of the plane into this sphere. Three cases are possible.
(i) $d < 0$. One can check that A, B and C are mapped into three big circles on the sphere. In this case our theorem follows from the theorem on altitudes of a spherical triangle (problem 8).
(ii) $d = 0$. Let us perform an inversion by a point D. Then our theorem is essentially the theorem on altitudes of euclidean triangle.
(iii) $d > 0$. One may check that the circles A, B and C map into three circles on the sphere, their centers lying on one spherical line. Denote this line by p. Consider a section of the sphere by the plane containing p, and perform an orthogonal projection from the sphere into this plane. Three circles constructed will map into three chords of p; denote these chords by KL, MN, PQ. Applying consequently our stereographic projection and orthogonal projection to C', we obtain some chord $P'Q'$. One may check that the line XY should pass through the intersection point of KL and MN, as well as through that of the lines tangent to p at P and Q. The chords $K'L'$ and $M'N'$ are defined similarly. We need to prove that the lines $K'L'$, $M'N'$, and $P'Q'$ are concurrent. One can prove this statement using the Ceva theorem in a trigopometric form. (This statement is in fact the theorem on altitudes in Lobachevskiy geometry, being considered in the Klein model.)
29. Let A_1, B_1, and C_1 be the *'centers'* of circles, corresponding to the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ (i.e. A_1 is the point of the sphere such that the plane that touches the sphere in this point is parallel to the circle A, and so on). Then the vectors $\overrightarrow{OA_1}$, $\overrightarrow{OB_1}$, and $\overrightarrow{OC_1}$ are parallel to $\vec{A}$, $\vec{B}$, and $\vec{C}$ respectively. Arguments similar to those in the solution of problem 4 show that the points A_1, B_1, C_1, and O lie in one plane, i.e. the 'centers' of the circles A, B, and C lie on one spherical line.
30. Let A_1 be the point such that $\overrightarrow{OA_1} = \vec{A}$. Denote by d_A the length of a tangent from A_1 to our sphere. Consider an arbitrary point C on the circle A. By the Pythagorean theorem for the triangle OA_1C, we obtain $OA_1^2 = OC^2 + A_1C^2$, i.e. $(\vec{A}, \vec{A}) = 1 + d_A^2$, QED.

31. This solution is obtained from the following (more general) formula for the angle γ between A and B:

$$\cos\gamma = \frac{(A,B)-1}{d_A d_B}.$$

Let us prove this formula. Let C be one of the intersection points of A and B. Let $\vec{C} = \overrightarrow{OC}$. Note that the vectors $[\vec{A},\vec{C}]$ and $[\vec{B},\vec{C}]$ are parallel to the lines passing through C and tangent to the circles A and B respectively. Hence

$$\cos\gamma = \frac{([\vec{A},\vec{C}],[\vec{B},\vec{C}])}{|[\vec{A},\vec{C}]|\cdot|\vec{B},\vec{C}|}.$$

Let us show that the right hand part coincides with the required expression. It is easy to check that $|[\vec{A},\vec{C}]| = |\vec{A}|\cdot|\vec{C}|\cdot\sin\angle(\vec{A},\vec{C}) = d_A$. Hence the denominator of the right hand part is equal to $d_A d_B$. Now we will rearrange the numerator using the identity from the second solution of problem 15. We have

$$([\vec{A},\vec{C}],[\vec{B},\vec{C}]) = (\vec{A},\vec{B})(\vec{C},\vec{C}) - (\vec{A},\vec{C})(\vec{B},\vec{C}) = (\vec{A},\vec{B}) - 1.$$

Substituting the expressions for the numerator and the denominator, we obtain the desired formula.

32. Assume that the vector $\vec{P} = \frac{[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]}{(\vec{A},\vec{B},\vec{C})}$ corresponds to a circle P (such a circle may not exist, in which case the given vector has no geometric interpretation). Notice that

$$(\vec{P},\vec{A}) = \frac{0+(\vec{A},[\vec{B},\vec{C}])+0}{(\vec{A},\vec{B},\vec{C})} = 1.$$

Then by Problem 31, the circle P is orthogonal to the circle A. Analogously, the circle P is orthogonal to the circles B and C. In other words, P is *the common perpendicular* to the circles A, B and C.

*Remark**. It is more natural to each circle to assign a pair $(\vec{A},h)$, where $\vec{A}$ is the unit vector orthogonal to the plane containing the circle, and h is the distance between the plane and the point O. In other words, to each circle we assign a quaternion. Denote this quaternion by the same letter as the circle itself. We assume that all the quaternions, obtained from the given one by the multiplication by a real number, correspond to the same circle. Then, for example, the quaternion $AB-BA$ corresponds to the line passing through the centers of the circles A and B, and the quaternion $ABC - ACB + BCA - BAC + CAB - CBA$ corresponds to the circle orthogonal to three circles A, B and C. And the commutator of a quadruple of quaternions $ABCD - ABDC + ...$ always equals to zero. This identity implies, for example, the following geometrical theorem:

Proposition. Let D be the center of the circumscribed circle of the triangle ABC, and let A', B' and C' be the centers of circumscribed circles of the triangles BCD, CAD and ABD respectively. Then the lines AA', BB' and CC' have a common point.

33. Assume that $\vec{A}-\vec{B}+\vec{C}-\vec{D}=0$. Consider the vector $\vec{P} = \frac{[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]}{(\vec{A},\vec{B},\vec{C})}$. Assume that it corresponds to some circle P. Then by Problem 32 the circle P is the common perpendicular to A, B and C. Notice that

$$(\vec{P},\vec{D}) = (\vec{P},\vec{A}-\vec{B}+\vec{C}) = 1-1+1 = 1,$$

hence by Problem 31 the circle P is orthogonal to the circle D. In other words, the four circles A, B, C and D have a common perpendicular.

34. First let us prove a Lemma:

Lemma. If there are nonzero numbers x, y and z, such that $x+y+z=0$ and $x\vec{A}+y\vec{B}+z\vec{C}=0$, then the circles A, B and C have a common point.

Proof of the lemma. The assumption of the lemma means that the ends of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ lie on one plane. Draw a plane π passing through this line and the point O. Let ω be the circle, which is the intersection of the sphere and the plane π. Let H be the pole of our line in the plane π with respect to the circle γ. Draw a line h passing through the point H and orthogonal to the plane π. Then one can check that all the circles A, B and C pass through the intersection point of the sphere and the line h.

Now let us solve our problem. Consider a circle C corresponding to the vector $\vec{C} = \frac{d_B}{d_B-d_A}\vec{A} - \frac{d_A}{d_B-d_A}\vec{B}$. Then by our lemma the circle C passes through both intersection points of the circles A and B. Denote by α, β and γ the angles between the pairs of circles B and C, C and A, A and B respectively. Then by the formula from the solution of the problem 31 we have

$$\cos\beta = \frac{(\vec{A},\vec{C})-1}{d_A d_C} = \frac{\frac{d_B}{d_B-d_A}(\vec{A},\vec{A}) - \frac{d_A}{d_B-d_A}(\vec{A},\vec{B}) - 1}{d_A d_C} = \frac{d_A d_B(1-\cos\gamma)}{d_C(d_B-d_A)}.$$

Writing down the analogous formula for $\cos\alpha$, we get $\cos\alpha = \cos\beta$. In other words, C is a circle passing through both intersection points of A and B and dividing the angle between this two circles into two equal parts ('*bisector*').

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