GL_n REPRESENTATION THEORY NOTES FOR 11-05

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Today we begin translating some of our results on GL_n representations into results about S_n representations. We'll also eventually (on Wednesday) set up Schur-Weyl Duality, which allows us to go back and forth between GL_n and S_d for any n, d.

Our first tool is the following.

Definition Let $|\lambda| = n$. The ***Specht module*** $\mathrm{Sp}(\lambda)$ is the $(1, \ldots, 1)$ weight-space of $V_\lambda(n)$. Note that a basis for the Specht module is given by the SSYTs of shape λ and entries $1, \ldots, n$, each occurring once. These are called the ***standard Young tableaux***. In particular,

$$\dim \mathrm{Sp}(\lambda) = \#\{\text{standard Young tableaux of shape } \lambda\}.$$

Warning: The tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$ is **not** standard, even though the rows and columns are strictly increasing. Standard means each entry $\{1, 2, \ldots, n\}$ is used once.

1. Main Theorem

Our main theorem is the following:

Theorem. As λ varies over the partitions of n, $\mathrm{Sp}(\lambda)$ varies over the irreducible representations of S_n, each occurring once.

Proof. First of all, $S_n \subset \mathrm{GL}_n$ as permutation matrices acting on $V_\lambda(n)$. If $\sigma \in S_n$ is a permutation, then it maps the $(a_1, \ldots, a_n)$ weight-space to the $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ weight-space.

This is a simple computation. Take a diagonal matrix $d = \mathrm{diag}(t_1, \ldots, t_n)$ and consider its action on σu, where u is in the $(a_1, \ldots, a_n)$ weight space. We have

$$\begin{aligned} d\sigma u = \sigma\left(\sigma^{-1} d\sigma\right) u = \sigma \,\mathrm{diag}(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)}) u = \sigma\left(t_{\sigma^{-1}(1)}^{a_1} t_{\sigma^{-1}(2)}^{a_2} \cdots t_{\sigma^{-1}(n)}^{a_n} \cdot u\right) \\ = \left(t_{\sigma^{-1}(1)}^{a_1} t_{\sigma^{-1}(2)}^{a_2} \cdots t_{\sigma^{-1}(n)}^{a_n}\right) \cdot \sigma u = \left(t_1^{a_{\sigma(1)}} t_2^{a_{\sigma(2)}} \cdots t_n^{a_{\sigma(n)}}\right) \cdot \sigma u \end{aligned}$$

So σu is in the weight space we claimed it was in.

In particular, we see that S_n acts on the $(1, \ldots, 1)$ weight space.

Now, we look inside the matrix coefficients ring $\mathbb{C}[z_{ij}]_{1 \le i,j \le n}$ for GL_n. Consider the terms which are degree 1 in every row and every column. These are weight $(-1, \ldots, -1)$ for the left GL_n action and weight $(1, \ldots, 1)$ for the right GL_n action.)

An obvious basis for this space is given by

$$\{z_{1\sigma(1)} \cdots z_{n\sigma(n)} : \sigma \in S_n\},$$

and we see that, as an $S_n \times S_n$ representation, this subspace is isomorphic to $\mathbb{C}[S_n]$. Now we know that, as a $\mathrm{GL}_n \times \mathrm{GL}_n$ representation, $\mathbb{C}[z_{ij}]$ has the decomposition from the weak Peter-Weyl theorem,

$$\mathbb{C}[z_{ij}] = \bigoplus_\lambda V_\lambda(n)^\vee \otimes V_\lambda(n).$$

By inspection, the only summands that contribute to the weight space we care about are those with $|\lambda| = n$. So, by restricting to the Specht modules contained in each $V_\lambda(n)$, we obtain

$$\bigoplus_{|\lambda|=n} \mathrm{Sp}(\lambda)^\vee \otimes \mathrm{Sp}(\lambda) \cong \mathbb{C}[S_n].$$

By a problem on an old problem set, (assuming each $\mathrm{Sp}(\lambda)^\vee \otimes \mathrm{Sp}(\lambda)$ is nonzero), this is automatically the decomposition of $\mathbb{C}[S_n]$ into irreducible $S_n \times S_n$ representations, and the $\mathrm{Sp}(\lambda)$ are automatically the irreducible S_n representations, each listed once. To finish the proof, note that $\mathrm{Sp}(\lambda)$ is always nonzero since there's certainly always at least one SYT of shape λ. $\square$

Note: All S_n representations are self-dual, since $\chi_{V^\vee}(\sigma) = \chi_V(\sigma^{-1})$, and σ is conjugate to σ^{-1} in S_n. So, $\mathrm{Sp}(\lambda)^\vee \cong \mathrm{Sp}(\lambda)$.

A fancier way of stating the above results is the following:

Corollary. Restriction to the $(1,\dots,1)$ weight-space gives an equivalence of categories

$$\big\{\mathrm{GL}_n \text{ polynomial irreps where } t\cdot\mathrm{Id} \text{ acts by } t^n\big\} \longrightarrow \big\{S_n \text{ representations}\big\}.$$

2. Examples

Here are four basic examples of Specht modules.

Example 1. We consider $\mathrm{Sp}\left(\square\!\cdot\!\cdot\!\cdot\!\square\right)$ with n boxes in one row. This is the subspace of $V_{\square\cdot\cdot\cdot\square}(n)$ of degree $(1,\dots,1)$:

$$V_{\square\cdot\cdot\cdot\square}(n) \cong \mathrm{Sym}^n\,\mathbb{C}^n = \mathbb{C}[z_1,\dots,z_n]_n,$$

so $\mathrm{Sp}\left(\square\!\cdot\!\cdot\!\cdot\!\square\right) = \mathbb{C}\cdot z_1\cdots z_n$, and S_n acts trivially. So, this is the trivial representation.

Example 2. We consider $\lambda = \begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}$ with n boxes in one column. Then $\mathrm{Sp}(\lambda)$ is the subspace of $V_\lambda(n)$ of degree $(1,\dots,1)$:

$$V_\lambda(n) \cong \bigwedge^n \mathbb{C}^n = \det(\cdot),$$

so the Specht module is one-dimensional, $\mathrm{Sp}(\lambda) = \mathbb{C}\cdot\Delta_{1\dots n}$, and S_n acts by permuting the columns in the determinant, which introduces a sign of $(-1)^\sigma$. So, this is the sign representation of S_n.

Example 1. We consider $\mathrm{Sp}\left(\begin{smallmatrix}\square\,\square\cdot\cdot\cdot\square\\ \square\end{smallmatrix}\right)$. This is the $\mathbb{C}$-span of the products

$$\Delta_{ij}\cdot\frac{z_{11}\cdots z_{1n}}{z_{1i}z_{1j}} = \det\begin{vmatrix} z_{1j} & z_{1j}\\ z_{2i} & z_{2j}\end{vmatrix}\cdot z_{11}\cdots\widehat{z_{1i}}\cdots\widehat{z_{1j}}\cdots z_{1n}.$$

The dimension is the number of SYTs of shape λ, which is $n-1$ (corresponding to the choices of the box on the second row), so there are various relations between the above generators. In particular, letting $p = z_{11}\cdots z_{1n}$ and $w_k = \frac{w_{2k}}{w_{1k}}$, we see that our generators above are given by $p(w_j - w_i)$, which leads to lots of relations. A nice way of putting it is:

$$\mathrm{Sp}\left(\begin{smallmatrix}\square\,\square\cdot\cdot\cdot\square\\ \square\end{smallmatrix}\right) \cong \{a_1w_1 + \cdots a_nw_n : \sum a_i = 0\} \subset \mathbb{C}^n,$$

which identifies it as the "standard representation" (the subrep of the "permutation representation" $\mathbb{C}^n$ that is orthogonal to the trivial subrep).

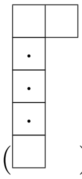
Example 4. Take the transpose of our last partition. Similarly to the above, we have

$$\mathrm{Sp}\left(\begin{smallmatrix}\square\,\square\\ \square\\ \vdots\\ \square\end{smallmatrix}\right) = \mathrm{Span}(z_{1k}\cdot\Delta_{1\cdots\widehat{k}\cdots n})_{k=1}^n.$$

This gives n generators, but there are only $n-1$ standard Young tableaux of this shape, so there is one relation. The relation is just the alternating sum:

$$\sum(-1)^k z_{1k}\cdot\Delta_{1\cdots\widehat{k}\cdots n} = 0.$$

In particular, we can write

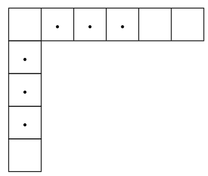
Sp() $\cong \mathbb{C}^n/(e_1 + \cdots + e_n)$,

where the S_n action is given by (the obvious action) $\otimes$ (the sign action).

These example provide evidence for the following equality (which is true):

$$\mathrm{Sp}(\lambda^T) = \mathrm{Sp}(\lambda) \otimes (\text{sign}).$$

Challenge. Compute the Specht module for



with $k+1$ boxes vertically and $n-k$ horizontally. (This is somehow "in between" the examples above.)

3. Next Time

On Wednesday, we'll show the following: with $n = \dim V$,

$$V^{\otimes d} = \bigoplus_{|\lambda|=d} \mathrm{Sp}(\lambda) \otimes V_\lambda(n)$$

as $S_d \times \mathrm{GL}_n$ representations.