NOTES FOR 14 APRIL 2011

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Where to read about what we've been doing:

- Picard group: early parts of Voisin Chapter 7. (But Voisin doesn't prove Kodaira or Lefschetz Vanishing.)
- Kodaira and Lefschetz Vanishing: Griffiths and Harris. See handout for their writeup on the Lefschetz hyperplane theorem.
- Kodaira Embedding: Chapter 7 of Voisin or Griffiths and Harris.
- Hypercohomology and Algebraic de Rham: Voisin Chapter 8

Hypercohomology

We want to get rid of smooth functions and stick to holomorphic ones, and use coherent vector bundles as our sheaves of choice. Our tool for this is hypercohomology.

Hypercohomology generalizes / relates to two notions of cohomology:

- The cohomology of a sheaf $H^j(X, \mathcal{F})$, where $\mathcal{F}$ is a sheaf of abelian groups on X.
- If we have a complex of sheaves
$$0 \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \ldots$$
(where $d^2 = 0$, but there is no exactness condition), then we can define
$$SH^k(\mathcal{F}^\bullet) := \frac{\mathcal{K}er(\mathcal{F}^k \to \mathcal{F}^{k+1})}{\mathcal{I}m(\mathcal{F}^{k-1} \to \mathcal{F}^k)}.$$
Then $SH^k(\mathcal{F}^\bullet) = 0$ if and only if $\mathcal{F}^\bullet$ is exact.

Definition. *Given two complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^0 & \longrightarrow & \mathcal{E}^1 & \longrightarrow & \mathcal{E}^2 & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \mathcal{F}^2 & \longrightarrow & \ldots \end{array}$$

of sheaves, a **quasi-isomorphism** $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ *is a collection of maps* $\mathcal{E}^k \to \mathcal{F}^k$*, which commute with* d *and induce isomorphisms* $SH^k(\mathcal{E}^\bullet) \to SH^k(\mathcal{F}^\bullet)$.

Example: what does a quasi isomorphism between $0 \to \mathcal{F} \to 0 \to 0 \to \ldots$ and $0 \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \ldots$ look like? The cohomology of the top sequence is $\mathcal{F}$ in the zeroth position and 0 elsewhere. Thus we need the second sequence to be exact except at $\mathcal{I}^0$, where it must have kernel $\mathcal{F}$. In other words, the map of complexes is a q.i. if and only if
$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \ldots$$
is a resolution.

Lemma. *Every complex of abelian sheaves has a quasi-isomorphism to a complex of injective sheaves.*

Definition/Theorem. *If* $\mathcal{E}^\bullet \overset{q.i.}{\to} \mathcal{I}^\bullet$*, then*
$$\mathbb{H}^k(\mathcal{E}^\bullet) = H^k(0 \to \mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \mathcal{I}^2(X) \to \ldots)$$
is the **hypercohomology** *of* $\mathcal{E}^\bullet$.

Key facts:

- This is well defined and functorial: given $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$, we get maps $\mathbb{H}^k(\mathcal{E}^\bullet) \to \mathbb{H}^k(\mathcal{F}^\bullet)$.
- If $\mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism, then $\mathbb{H}^k(\mathcal{A}^\bullet) \overset{\cong}{\to} \mathbb{H}^k(\mathcal{B}^\bullet)$.

- If $\mathcal{E}^\bullet$ is a complex of acyclic sheaves (e.g. if $\mathcal{E}^k$ is injective), then
$$\mathbb{H}^k(\mathcal{E}^\bullet) = H^\bullet(0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to ...).$$
Hence we can compute hypercohomology with acyclic sheaves.
- If we have three complexes
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \mathcal{A}^2 & \longrightarrow & ... \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B}^0 & \longrightarrow & \mathcal{B}^1 & \longrightarrow & \mathcal{B}^2 & \longrightarrow & ... \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^0 & \longrightarrow & \mathcal{C}^1 & \longrightarrow & \mathcal{C}^2 & \longrightarrow & ... \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$
with exact columns, we get the long exact sequence
$$0 \to \mathbb{H}^0(\mathcal{A}^\bullet) \to \mathbb{H}^0(\mathcal{B}^\bullet) \to \mathbb{H}^0(\mathcal{C}^\bullet) \to \mathbb{H}^1(\mathcal{A}^\bullet) \to ....$$

Čech form of $\mathbb{H}$

Let $\mathcal{E}^\bullet$ be a complex of sheaves of abelian groups. Let $U_\bullet$ be a cover of X such that every $\mathcal{E}^p$ is acyclic on every $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_q}$ (for now assume such an open cover exists). Then we get a double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{i_0,i_1,i_2} \mathcal{E}^0(U_{i_0} \cap U_{i_1} \cap U_{i_2}) & \longrightarrow & \bigoplus_{i_0,i_1,i_2} \mathcal{E}^1(U_{i_0} \cap U_{i_1} \cap U_{i_2}) & \longrightarrow & \bigoplus_{i_0,i_1,i_2} \mathcal{E}^2(U_{i_0} \cap U_{i_1} \cap U_{i_2}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{i_0,i_1} \mathcal{E}^0(U_{i_0} \cap U_{i_1}) & \longrightarrow & \bigoplus_{i_0,i_1} \mathcal{E}^1(U_{i_0} \cap U_{i_1}) & \longrightarrow & \bigoplus_{i_0,i_1} \mathcal{E}^2(U_{i_0} \cap U_{i_1}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{i_0} \mathcal{E}^0(U_{i_0}) & \longrightarrow & \bigoplus_{i_0} \mathcal{E}^1(U_{i_0}) & \longrightarrow & \bigoplus_{i_0} \mathcal{E}^2(U_{i_0}) & \longrightarrow & \cdots
\end{array}$$

Whenever we have a double complex, we can collapse it to a single complex:

$$0 \to \bigoplus \mathcal{E}^0(U_{i_0}) \to \begin{array}{c} \bigoplus \mathcal{E}^0(U_{i_0} \cap U_{i_1}) \\ \bigoplus \\ \bigoplus \mathcal{E}^1(U_{i_0}) \end{array} \to \begin{array}{c} \bigoplus \mathcal{E}^0(U_{i_0} \cap U_{i_1} \cap U_{i_2}) \\ \bigoplus \\ \bigoplus \mathcal{E}^1(U_{i_0} \cap U_{i_1}) \\ \bigoplus \\ \bigoplus \mathcal{E}^2(U_{i_0}) \end{array} \to ....$$

There is a sign twist: In ever square of the original diagram, one of the 4 maps should get a -1 (conventions differ as to where). Then $\mathbb{H}^k(\mathcal{E}^\bullet)$ is the cohomology of this "hyper-Čech" complex.

Example: Let W be our favorite hyperelliptic curve, $W = W_1 \cup W_2$, with

$$y_1^2 = x_1^{2g+1} + \cdots + x_1$$
$$y_2^2 = x_2^{2g+1} + \cdots + x_2$$

(with the coefficients reversed in the two equations).

Let's look at $\mathbb{H}^\bullet(W, \mathcal{O} \stackrel{\partial}{\to} \mathcal{H}^1)$:

$$\begin{array}{ccc} \mathcal{O}(W_1 \cap W_2) & \xrightarrow{-1} & \mathcal{H}^1(W_1 \cap W_2) \\ \uparrow & & \uparrow \\ \mathcal{O}(W_1) \oplus \mathcal{O}(W_2) & \longrightarrow & \mathcal{H}^1(W_1) \oplus \mathcal{H}^1(W_2) \end{array}$$

Then we $\mathbb{H}^1$ consists of triples

$$\frac{\{(f, \omega_1, \omega_2) \in \mathcal{O}(W_1 \cap W_2) \times \mathcal{H}^1(W_2) \times \mathcal{H}^1(W_2) : \omega_1 - \omega_2 = df\}}{\{(g_1 - g_2, dg_1, dg_2) : (g_1, g_2) \in \mathcal{O}(W_1) \times \mathcal{O}(W_2)\}}$$

(which we can compute explicitly if we go back to old problem sets). Note that global 1-forms are a subspace: if $\omega \in H^0(\mathcal{H}^1)$, then $(0, \omega, \omega) \in \mathbb{H}^1$. Conversely, if we quotient by global 1-forms, we get a map to $H^1(\mathcal{O})$ given by $(f, \omega_1, \omega_2) \mapsto f$. Hence we have a sequence

$$H^0(\mathcal{H}^1) \to \mathbb{H}^1(\mathcal{O} \to \mathcal{H}^1) \to H^1(\mathcal{O}).$$

Theorem. *For any complex manifold X,*

$$H^k_{top}(X, \mathbb{C}) \cong \mathbb{H}^k(\mathcal{O} \stackrel{\partial}{\to} \mathcal{H}^1 \stackrel{\partial}{\to} \mathcal{H}^2 \stackrel{\partial}{\to} \ldots).$$

Proof.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{H}^1 & \longrightarrow & \mathcal{H}^2 & \longrightarrow & \ldots \end{array}$$

is a quasi-isomorphism. □

So hypercohomology has packaged all the work into convenient notation. The only work remaining is to prove that the above is a quasi-isomorphism (true by Dolbeault's Lemma), and since we know hypercohomology is invariant under quasi-isomorphism, this must compute the topological cohomology.

Let us call $\mathcal{H}^\bullet(\mathcal{O} \to \mathcal{H}^1 \to \mathcal{H}^2 \to \ldots)$ "analytic de Rham" (in contrast with "algebraic de Rham" which we will discuss next time).

What happens to Dolbeault?

We have the double complex

$$\begin{array}{ccccccc} \ldots & & \ldots & & \ldots & & \\ \bar\partial \uparrow & & \bar\partial \uparrow & & \bar\partial \uparrow & & \\ \Omega^{0,2}(X) & \xrightarrow{\partial} & \Omega^{1,2}(X) & \xrightarrow{\partial} & \Omega^{2,2}(X) & \xrightarrow{\partial} & \ldots \\ \bar\partial \uparrow & & \bar\partial \uparrow & & \bar\partial \uparrow & & \\ \Omega^{0,1}(X) & \xrightarrow{\partial} & \Omega^{1,1}(X) & \xrightarrow{\partial} & \Omega^{2,1}(X) & \xrightarrow{\partial} & \ldots \\ \bar\partial \uparrow & & \bar\partial \uparrow & & \bar\partial \uparrow & & \\ \Omega^{0,0}(X) & \xrightarrow{\partial} & \Omega^{1,0}(X) & \xrightarrow{\partial} & \Omega^{2,0}(X) & \xrightarrow{\partial} & \ldots \end{array}$$

(the squares already anticommute, so the minus sign is already "built in").

The corresponding single complex is just

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots,$$

the de Rham complex.

$\mathbb{H}^\bullet(\mathcal{O} \to \mathcal{H}^1 \to \mathcal{H}^2 \to \ldots)$ is computed from ordinary cohomology of the complex of smooth bundles. By Hodge theory, if X is compact Kähler, then cohomology is represented by the harmonic functions in this double complex.

If we do not have access to the double complex, though, how can we discuss the Hodge decomposition? Define

$$F^pH^k := \mathbb{H}^k(0 \to 0 \to ... \to 0 \to \mathcal{H}^p \to \mathcal{H}^{p+1} \to \mathcal{H}^{p+2} \to ...)$$

(where the $\mathcal{H}^p$ is in position p). In particular, F^0H^k is analytic de Rham cohomology $\mathbb{H}^\bullet(\mathcal{O} \to \mathcal{H}^1 \to \mathcal{H}^2 \to ...)$. This is the definition we will generalize, but we will now discuss two other ways to think about it (which will not generalize):

- $0 \to \mathcal{Z}^p \to \mathcal{H}^p \to \mathcal{H}^{p+1} \to ...$ is a resolution of $\mathcal{Z}^p := \{\text{closed } (p,0)\text{-forms}\}$. So $F^pH^k(X) \cong H^{k-p}(X, \mathcal{Z}^p)$. The shift of indices comes up because the complex starts at position p, not 0.
- We have a Dolbeault complex for F^pH^k: erase the first p columns of the standard one. If X is compact Kähler, then Hodge decomposition gives

$$F^pH^k(X) \cong \bigoplus_{\substack{p' \geq p \\ p'+q'=k}} H^{p',q'}(X)$$

We always (even without the compact Kähler assumption) have

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & ... & \to & 0 & \to & \mathcal{H}^{p+1} & \to & \mathcal{H}^{p+2} & \to & \mathcal{H}^{p+3} & \to & ... \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & ... & \to & 0 \to \mathcal{H}^p & \to & \mathcal{H}^{p+1} & \to & \mathcal{H}^{p+2} & \to & \mathcal{H}^{p+3} & \to & ... \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & ... & \to & 0 \to \mathcal{H}^p & \to & 0 & \to & 0 & \to & 0 & \to & ... \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}.$$

To remember which is the first and which the third row, think about how to make the squares commute.

This gives us the long exact sequence

$$... \to H^{k-p-1}(X, \mathcal{H}^p) \to F^{p+1}H^k(X) \to F^pH^k(X) \to H^{k-p}(X, \mathcal{H}^p) \to F^{p+1}H^{k+1}(X) \to ....$$

When X is compact Kähler, this breaks into short exact sequences

$$0 \to F^{p+1}H^k(X) \to F^pH^k(X) \to H^{k-p}(X, \mathcal{H}^p) \to 0$$

(proved by representing everything as harmonic classes).

So, in the compact Kähler case, we have injections

$$F^kH^k \subset F^{k-1}H^k \subset ... \subset F^2H^k \subset F^1H^k \subset F^0H^k = H^k_{\text{top}}(X, \mathbb{C}),$$

the "Hodge filtration," and

$$F^pH^k/F^{p+1}H^k = H^{k-p}(X, \mathcal{H}^p).$$

If complex conjugation makes sense, we can recover the direct sum decomposition by

$$H^{p,q} = F^pH^{p+q} \cap \overline{F^pH^{p+q}}.$$

Think of $H^k(X, \mathbb{C})$ as $H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and take the conjugate. However, we do not have such things in characteristic p, and even in characteristic zero, there is still a sense in which the filtration is more natural than the direct sum decomposition.

Recall from a problem set:

Let $H = \{x + iy : y > 0\}$. Let $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$ for $\tau \in H$. We have a family

$$\begin{array}{c} E \\ \downarrow \\ H \end{array}$$

whose fiber over τ is the genus 1 curve E_τ. Topologically, this is a trivial family. We can look at the vector bundle V whose fiber over τ is $H^1(E_\tau, \mathbb{C})$. This is a trivial vector bundle - even canonically trivial. It has natural trivialization e_1^*, e_2^* where $e_1^* \in H^1(E_\tau, \mathbb{Z})$ is 1 on $\mathbb{R}/\mathbb{Z}$ and 0 on $\mathbb{R}\tau/\mathbb{Z}\tau$, and $e_2^* \in H^1(E_\tau, \mathbb{Z})$ is 0 on $\mathbb{R}/\mathbb{Z}$ and 1 on $\mathbb{R}\tau/\mathbb{Z}\tau$. Hence we can define holomorphic sections in terms of this natural trivialization. We computed

$$H^{1,0}(E_\tau) = \mathbb{C} \cdot (e_1^* + \tau e_2^*).$$

In our new language, F^1H^1 is a holomorphic subbundle of H^1. But $H^{0,1}(E_\tau) = e_1^* + \overline{\tau} e_2^*$ is *not* a holomorphic subbundle. So we see that the filtration is natural, not the direct sum.