Problem Set 5 – due Wednesday, October 10

See the course website for policy on collaboration.

1. Let X be a Zariski closed subset of $\mathbb{A}^n$. Let f be a regular function on X. The open set $\{f \neq 0\}$ is denoted $D(f)$; an open set of this form is called a distinguished open. Let $Y \subset \mathbb{A}^{n+1} = \{(x,t) : x \in X,\ ft = 1\}$.
   (a) Show that Y is isomorphic to $D(f)$ by giving regular maps in each direction.
   (b) Conclude that every regular function on $D(f)$ is of the form $\frac{g}{f^N}$ for some regular function g on X and some nonnegative integer N.
   (c) Check that basic open sets are a basis for the Zariski topology on X.

2. Let $B = \operatorname{MaxSpec} A$ and let X be Zariski closed in $B \times \mathbb{P}^n$. Show that $X = Z(I)$ for some homogenous ideal $I \subset A[x_0, x_1, \ldots, x_n]$.

3. Let A be a commutative ring and I and J ideals. Then $[I : J]$ is defined by
$$[I : J] := \{f \in A : fj \in I \text{ for all } j \in J\}.$$
The ideal $[I : J^\infty]$, called the ***saturation of I with respect to J*** is defined to be
$$[I : J^\infty] = \bigcup_{n=0}^{\infty} [I : J^n].$$
Let $\overline{S}$ denote the Zariski closure of S.
   (a) Let I and $J \subset k[x_1, \ldots, x_n]$ be radical ideals, with $I = I(X)$ and $J = I(Y)$. Show that $[I : J]$ is radical, $[I : J] = [I : J^\infty]$ and $[I : J] = I(\overline{X \setminus (X \cap Y)})$.
   (b) Let I and $J \subset k[x_1, \ldots, x_n]$ be ideals, not necessarily radical, with $X = Z(I)$ and $Y = Z(J)$. Show that $Z([I : J^\infty]) = \overline{X \setminus (X \cap Y)}$.

4. Let $U \subset \mathbb{P}^2$ be the complement of the conic $p^2 + q^2 + r^2 = 0$. In this problem, we will show that U is isomorphic to an affine variety. A similar proof shows $\mathbb{P}^N \setminus \{F = 0\}$ is affine for any homogenous polynomial F.
Embed $\mathbb{P}^2$ into $\mathbb{P}^5$ by $\phi : (p : q : r) \mapsto (p^2 : pq : pr : q^2 : qr : r^2)$. You found that $\phi(\mathbb{P}^2)$ is closed in $\mathbb{P}^5$, with equations $ux = v^2$, $uy = w^2$, $xz = y^2$, $uy = vw$, $vz = wy$ and $wx = vy$.
   (a) Show that $\phi(U)$ lies in a linear chart of $\mathbb{P}^5$, and is closed in that linear chart.
   (b) Give explicit generators and relations for the ring of regular functions on U.

5. In this problem, we will work with $\mathbb{P}^9$ and label the homogeneous coordinates as a_{ijk} for $0 \leq i, j, k,\ i + j + k = 3$. We think of (a_{ijk}) as encoding the cubic curve $\sum a_{ijk} x^i y^j z^k$ in $\mathbb{P}^2$. Show that the set of cubics which factor as (linear)(quadratic) is Zariski closed in $\mathbb{P}^9$.

6. I used to believe that, if $\phi : \mathbb{A}^m \to \mathbb{A}^n$ was given by homogenous polynomials all of the same degree, then $\phi(\mathbb{A}^m)$ is Zariski closed. This is false! Give a counterexample.

7. In the proof of his second statement of Noether normalization (Theorem I.5.4.10), Shavarevich implicitly assumes the following statement. Prove it.

   > Let $X \subsetneq \mathbb{A}^n$ be Zariski closed in $\mathbb{A}^n$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Then there is some point of $\mathbb{P}^n \setminus \mathbb{A}^n$ which is not in $\overline{X}$.