Problem Set 4 – due October 3

See the course website for policy on collaboration.

1. Two conics should intersect at 4 points. So, what are the 4 points of $\mathbb{P}^2$ where the circles $(x-3)^2+y^2=25$ and $(x+3)^2+y^2=25$ meet?
2. Let X and Y be topological spaces and $\phi : X \to Y$ a surjective map. If X is irreducible, show that Y is also irreducible.
3. Let X be a topological space. Show that X is irreducible if and only if every nonempty open subset of X is dense.
4. Let $X = Z(y(x^2-y)) \subset \mathbb{A}^2$. Define $f : X \to k$ by

$$f(x,y) = \begin{cases} x & y = x^2 \\ 0 & y = 0 \end{cases}.$$

   Show that f is not a regular function.

5. Let X and Y be Zariski closed in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. We'll write π_X and π_Y for the projections $X \times Y \to X$ and $X \times Y \to Y$ respectively.
   (a) Show that $X \times Y$ is Zariski closed in $\mathbb{A}^m \times \mathbb{A}^n$.
   (b) Show that $\mathcal{O}_{X\times Y}$ is generated by the pullbacks of $\mathcal{O}_X$ and $\mathcal{O}_Y$ along π_X and π_Y respectively.
6. Prove that the only regular functions $\mathbb{P}^1 \to k$ are the constants.
7. Map $\mathbb{P}^2$ to $\mathbb{P}^5$ by $\phi : (p:q:r) \mapsto (p^2 : pq : pr : q^2 : qr : r^2)$. We write $(u:v:w:x:y:z)$ for the coordinates on $\mathbb{P}^5$.
   (a) Show that ϕ is injective.
   (b) Show that the image of ϕ is closed, and give explicit homogenous equations for the image.
   (c) Show that the inverse map $\phi(\mathbb{P}^2) \to \mathbb{P}^2$ is regular.