PROBLEM SET 1 : DUE FRIDAY, JANUARY 18

See the course website for homework policy.

Problem 1 Let G be the group of rotational symmetries of the cube.
(a) Describe the stabilizer of a face of the cube.
(b) Describe the stabilizer of a vertex of the cube.
(c) What is $|G|$?
(d) (**Harder**) Show that $G \cong S_4$.

Problem 2 Let G be a finite group and X a finite set on which G acts. For $g \in G$, let X^g be the number of elements of X fixed by g. Show that the number of orbits of G acting on X is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Problem 3 Let G be a finite group. For $g \in G$, let $\operatorname{ord}(g)$ be the order of the element g. For g and $h \in G$, we define $g \equiv h$ if $h = g^k$ for some k with k relatively prime to $\operatorname{ord}(g)$.
(a) Show that $\equiv$ is an equivalence relation.
(b) For g and $h \in G$, define $g \approx h$ if there is some g' which is conjugate to g with $g' \equiv h$. Show that $\approx$ is an equivalence relation.
(c) Let X be a finite set on which G acts, and suppose that $g \approx h$ for some g and h in G. For every integer i, show that the number of orbits of size i for g acting on X is the same as the number of orbits of size i for h acting on X.
(d) (**Harder**) Let G be a finite group and suppose that $g \not\approx h$ for some g and h in G. Construct a finite set X on which G acts so that the orbits of g on X have different sizes than the orbits of h.

Problem 4 Let G be a finite group. Let H be a subgroup of G with $n = [G : H]$.
(a) Show that $[G : \bigcap_{x \in G} xHx^{-1}]$ divides $n!$.
(b) Suppose that n is the smallest prime dividing $|G|$. Show that H is normal in G.

Problem 5 Let G be a group and let X be a finite set on which G acts transitively and with trivial stabilizer.
(a) Let H be the group of bijections $\phi : X \to X$ such that $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$. Show that $H \cong G$.
(b) To what extent is your isomorphism in (a) uniquely determined by the data of G and X?

Problem 6 Which of the following short exact sequences are semi-direct:
(a) $0 \to A_5 \to S_5 \to \mathbb{Z}/2 \to 0$?
(b) $0 \to \mathbb{Z}/2 \to \mathbb{Z}/6 \to \mathbb{Z}/3 \to 0$?
(c) $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 0$?
(d) $0 \to \{1, i, -1, -i\} \to Q \to \mathbb{Z}/2 \to 0$, where Q is the eight element subgroup of the quaternions consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$?

Problem 7 Let G, A, B and C be finite groups. Consider the following three statements:

(1) There are subgroups $P \subset Q \subset G$, with P normal in Q and Q normal in G, such that $P \cong A$, $Q/P \cong B$ and $G/Q \cong C$.
(2) There is a normal subgroup Q of G such that we have short exact sequences $0 \to Q \to G \to C \to 0$ and $0 \to A \to Q \to B \to 0$.
(3) There is a quotient group R of G such that we have short exact sequences $0 \to A \to G \to R \to 0$ and $0 \to B \to R \to C \to 0$.

(a) Two of these statements are logically equivalent. Which two?
(b) Show that the remaining statement implies the other two.
(c) (**Harder**) Give an example of a group G for which the two statements of (a) are true but the remaining statement is false.