Section A

1. For each $a \in A$, let $r_a : A \to A$ be the left A-linear map given by

$$r_a(b) = ba \quad \text{for all} \quad b \in A.$$

Prove that the map $r : A^{\text{op}} \to \text{End}_A(A)$ given by $r(a) = r_a$ is an isomorphism of rings.

Solution. It is clear that r is additive. If $r(a) = 0$ then $a = 1a = r(a)(1) = 0$ so r is injective. Let $\omega : A \to A$ be left A-linear and let $a := \omega(1)$. Then $\omega(b) = \omega(b \cdot 1) = b \cdot \omega(1) = ba = r(a)(b)$ for all $b \in A$, so $\omega = r(a)$. Hence r is surjective. Finally, given $a, b, c \in A$ we have $r(a \star b)(c) = r(ba)(c) = c(ba) = (cb)(a) = r(b)(c)a = r(a)(r(b)(c)) = (r(a) \circ r(b))(c)$. So $r(a \star b) = r(a) \circ r(b)$ and r is a ring homomorphism.

2. (a) Suppose that $|G| \neq 0$ in k. Prove that $e := \frac{1}{|G|} \sum_{g \in G} g$ is a central idempotent in kG.

(b) Let $G = C_3 = \langle x \rangle$ be a cyclic group of order 3. Suppose that $\text{char}(k) \neq 3$ and that k contains a primitive cube root of unity ω. Find an explicit isomorphism of k-algebras $k \times k \times k \xrightarrow{\cong} kC_3$.

Solution. (a) For any $x \in G$ we have $|G|ex = \sum_{g \in G} gx = \sum_{h \in G} h = |G|e$. So $ex = e$ for all $x \in G$ and similarly $xe = e$ for all $x \in G$. So e is central. But then $|G|e^2 = e \sum_{g \in G} g = \sum_{g \in G} eg = |G|e$, so $e^2 = e$.

(b) There is an isomorphism $\varphi : kC_3 \to k \times k \times k$ given by $\varphi(f) = (f(1), f(\omega), f(\omega^2))$ for $f(x) \in kC_3$. We want to compute φ^{-1} explicitly. Let $e_0 = \varphi^{-1}(1, 0, 0)$, $e_1 = \varphi^{-1}(0, 1, 0)$, $e_2 = \varphi^{-1}(0, 0, 1)$ and write $e_i = p_i(x)$ for some quadratic polynomial p_i. Then $p_i(\omega^j) = \delta_{i,j}$. This leads to a system of 3×3 linear equations whose solution is

$$e_0 = \frac{1 + x + x^2}{3}, \quad e_1 = \frac{1 + \omega^2 x + \omega x^2}{3} \quad \text{and} \quad e_2 = \frac{1 + \omega x + \omega^2 x^2}{3}.$$

Thus $\{e_0, e_1, e_2\}$ is a set of pairwise orthogonal idempotents such that $e_0 + e_1 + e_2 = 1$. The required isomorphism φ^{-1} is given by $(a_0, a_1, a_2) \mapsto a_0 e_0 + a_1 e_1 + a_2 e_2$.