Section A

1. Suppose that k is the finite field $\mathbb{F}_p$ and let $G = \langle g \rangle$ be the cyclic group of order p. Let $\rho : G \to \mathrm{GL}_2(k)$ be the matrix representation given by

$$\rho(g^i) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \text{for each} \quad i \in \mathbb{Z}$$

Let $\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for $V = k^2$. Show that the G-stable subspace $\langle v_1 \rangle$ has no G-stable complement in $V = \langle v_1, v_2 \rangle$.

Solution. If W is a G-stable complement for $\langle v_1 \rangle$ in $V = \langle v_1, v_2 \rangle$, then $\dim W = 1$. The composition $W \to V/\langle v_1 \rangle$ of the homomorphisms of G-representations $W \hookrightarrow V$ and $V \twoheadrightarrow V/\langle v_1 \rangle$ is then an isomorphism of G-representations. But $V/\langle v_1 \rangle$ is the trivial representation, so W must be spanned by a G-fixed vector. Suppose that $u = \lambda_1 v_1 + \lambda_2 v_2$ is fixed by G. Then $g \cdot u = \lambda_1 v_1 + \lambda_2 (v_1 + v_2) = u + \lambda_2 v_1 = u$ forces $\lambda_2 = 0$. Hence $u = \lambda_1 v_1 \in \langle v_1 \rangle$. But then $W \cap U$ cannot be the zero subspace. So, $\langle v_1 \rangle$ does not admit any G-stable complement in V.

See Example 1.17.

2. Let G be a finite group. For each conjugacy class C in G, define its *conjugacy class sum* to be $\widehat{C} := \sum_{x\in C} x \in kG$. Prove that the conjugacy class sums form a basis for $Z(kG)$.

   Solution. Let $y \in G$. Then $y\widehat{C}y^{-1} = \sum_{x\in C} yxy^{-1}$. But the map $x \mapsto yxy^{-1}$ is a permutation of C. So, $y\widehat{C}y^{-1} = \widehat{C}$ for all $y \in G$. Hence $y\widehat{C} = \widehat{C}y$ for all $y \in G$. Since G spans kG, $\widehat{C}$ is central in kG: $\widehat{C} \in Z(kG)$.

   Let $z = \sum_{g\in G} a_g g$ be a central element of kG. Then for all $x \in G$, $z = xzx^{-1} = \sum_{g\in G} a_g xgx^{-1} = \sum_{h\in G} a_{x^{-1}hx} h$. Comparing coefficients, we get $a_g = a_{x^{-1}gx}$ for all $g, x \in G$. So, the function $G \to k, g \mapsto a_g$, is constant on conjugacy classes. Let $C_1, \cdots, C_s$ be the conjugacy classes of G, pick $g_i \in C_i$ and let $a_i := a_{g_i}$. Then $z = \sum_{i=1}^{s} a_i \sum_{g\in C_i} g = \sum_{i=1}^{s} a_i \widehat{C_i}$ shows $\{\widehat{C_1}, \cdots, \widehat{C_s}\}$ spans the centre $Z(kG)$. If $\sum_{i=1}^{s} \lambda_i \widehat{C_i} = 0$, then look at the coefficient of g_i to see $\lambda_i = 0$ for all i.

3. Let $g \in \mathrm{GL}(V)$ be an element of finite order and suppose that k is algebraically closed. Prove that g is diagonalisable whenever $\mathrm{char}(k) = 0$.
   Does this result also hold for fields of positive characteristic?

   Solution. Let n be the order of g, so that g is a root of the polynomial $t^n - 1$. Let $m_g(t)$ be the minimal polynomial of g. Then $m_g(t)$ divides $t^n - 1$. Since k is algebraically closed, $t^n - 1$ splits into a product of linear factors. Since $\mathrm{char}(k) = 0$, all of these factors are distinct. Hence $m_g(t)$ also splits into a product of distinct linear factors. So, by the Primary Decomposition Theorem, the linear map $g : V \to V$ must be diagonalisable.

   The result does *not* hold for fields of positive characteristic, as the example $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{F}_p})$ shows.