The Closed Range Theorem

by
Francis J. Narcowich
November, 2013

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and the compact operators on $\mathcal{H}$, respectively.

Our aim here is to prove that the range of an operator of the form $L = I - \lambda K$, where K is compact, is closed. We will begin with this lemma.

Lemma 0.1. *Let $K \in \mathcal{C}(\mathcal{H})$, $\lambda \in \mathbb{C}$, and $L = I - \lambda K$. If $f \in \overline{R(L^*)}$, then there is a constant $c > 0$, independent of f, such that $\|Lf\| \geq c\|f\|$.*

Proof. If not, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \overline{R(L^*)}$ such that $\|f_n\| = 1$ and $\|Lf_n\| \to 0$ as $n \to \infty$. Note that $Lf_n = f_n - \lambda K f_n$, so $f_n = \lambda K f_n + Lf_n$. Since the f_n's are bounded and K is compact, we may choose a subsequence $\{f_{n_k}\}$ such that $\{Kf_{n_k}\}$ is convergent. Thus, $\lim_{k\to\infty} f_{n_k} = \lambda \lim_{k\to\infty} Kf_{n_k} + \lim_{k\to\infty} Lf_{n_k}$. Both terms on the right are convergent, so f_{n_k} is also convergent. Let $f = \lim_{k\to\infty} f_{n_k}$. By the previous equation, we have that $f = \lambda K f$, so $Lf = 0$ – i.e., $f \in N(L)$. In addition, because $\overline{R(L^*)}$ is closed, $f \in \overline{R(L^*)}$. Since these spaces are orthogonal, f is orthogonal to itself and, consequently, $f = 0$. However, $\lim_{n\to\infty} \|f_n\| = 1 = \|f\|$. This is a contradiction. $\square$

Theorem 0.2 (Closed Range Theorem). *If $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then the range of the operator $L = I - \lambda K$ is closed.*

Proof. We want to show that if there is sequence $\{g_n\} \subset R(L)$ such that $g_n \to g$, then $g = Lf$ for some $f \in \mathcal{H}$. To begin, note that the solution f_n to $g_n = Lf_n$ is not unique if $N(L) \neq \{0\}$. Since $\mathcal{H} = N(L) \oplus \overline{R(L^*)}$, with the two spaces being orthogonal, we may make a unique choice by requiring that f_n be in $\overline{R(L^*)}$. Lemma 0.1 then implies that $\|g_n - g_m\| = \|L(f_n - f_m)\| \geq c\|f_n - f_m\|$. Because the convergent sequence $\{g_n\}$ is Cauchy, this inequality also implies that $\{f_n\}$ is Cauchy. Thus, $\{f_n\}$ is convergent to some $f \in \mathcal{H}$. It follows that $g = \lim_{n\to\infty} Lf_n = Lf$, so $g \in R(L)$. $\square$

The Closed Range Theorem allows us to apply the Fredholm alternative to equations of the form $u - \lambda K u = f$. Thus, we have the following result, which applies, for example, to finite rank or Hilbert-Schmidt operators.

Corollary 0.3. *Let $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. The equation $u - \lambda K u = f$ has a solution if and only if $f \in N(I - \bar{\lambda} K^*)^{\perp}$.*

Definition 0.4. *A subset S of $\mathcal{H}$ is said to be* compact *if and only if it is closed and every sequence in S has a convergent subsequence. S is said to be precompact if its closure is compact.*

Proposition 0.5. *Here are some important properties of compact sets.*

1. *Every compact set is bounded.*

2. *A subset S of $\mathcal{H}$ is precompact if and only if every sequence has a convergent subsequence (the limit is in $\bar{S}$).*

3. *Let $\mathcal{H}$ be finite dimensional. Every closed, bounded subset of $\mathcal{H}$ is compact.*

4. *In an infinite dimensional space, closed and bounded is not enough.*

Proof. For property 1, assume that S is an unbounded compact set. Since S is unbounded, we may select a sequence $\{v_n\}_{n=1}^{\infty}$ such that $\|v_n\| \to \infty$ as $n \to \infty$. Since S is compact, this sequence will have a convergent subsequence, say $\{v_k\}_{k=1}^{\infty}$, which will still be unbounded. This sequence is Cauchy, so there is a positive integer K for which $\|v_\ell - v_m\| \leq 1/2$ for all $\ell, m \geq K$. Fix ℓ and note that by the triangle inequality $\|v_m\| \leq 1/2 + \|v_\ell\|$. Now, the right side is bounded, because ℓ is fixed. However, $\|v_m\| \to \infty$ as $m \to \infty$. This is a contradiction, so S must be bounded.

For property 4, let $S = \{f \in \mathcal{H} \colon \|f\| \leq 1\}$. Every o.n. basis $\{\phi_n\}_{n=1}^{\infty}$ is in S. However, for such a basis $\|\phi_m - \phi_n\| = \sqrt{2}$, $n \neq m$. Again, this means there are no Cauchy subsequences in $\{\phi_n\}_{n=1}^{\infty}$, and consequently, no convergent subsequences. Thus, S is not compact.

For property 2, ($\Longrightarrow$): Suppose S is precompact and let (x_n) be a sequence in S. Then (x_n) is a sequence in $\overline{S}$, which is compact. Hence there exists a subsequence (x_{n_k}) of (x_n) and a point $x \in \overline{S}$ such that $x_{n_k} \to x$.

($\Longleftarrow$): Suppose every subsequence in S has a convergent subsequence. We show that $\overline{S}$ is compact. To do so, we show that every sequence in $\overline{S}$ has a subsequence which converges to a point in $\overline{S}$. Let (x_n) be a sequence in $\overline{S}$. Since every element in $\overline{S}$ is the limit of a sequence in S, for each n there exists a sequence $(y_{n,m})_m$ in S such that $y_{n,m} \to x_n$. Thus, for each n, there exists a natural number M_n such that

$$\|y_{n,m} - x_n\| < 1/n, \quad \text{for all } m \geq M_n.$$

Since the diagonal sequence $(y_{n,M_n})_n$ is a sequence in S, our hypothesis gives a subsequence $(y_{n_k,M_{n_k}})_k$ of it that converges to a point y. We show that

$x_{n_k} \to y$. Let $\varepsilon > 0$. Choose a natural number N such that $1/N < \varepsilon/2$ and

$$\|y_{n_k,M_{n_k}} - y\| < \varepsilon/2, \quad \text{for all } k \geq N.$$

Then if $k \geq N$, we have $n_k \geq k \geq N$ and hence

$$\|x_{n_k} - y\| \leq \|x_{n_k} - y_{n_k,M_{n_k}}\| + \|y_{n_k,M_{n_k}} - y\| < 1/n_k + \varepsilon/2 < \varepsilon.$$

Since $\overline{S}$ is closed, we also have that $y \in \overline{S}$. This completes the proof. □

1 Compact Operators

Definition 1.1. *Let $K : \mathcal{H} \to \mathcal{H}$ be linear. K is said to be compact if and only if K maps bounded sets into precompact sets. Equivalently, if $\{v_n\}_{n=1}^{\infty}$ is bounded, then the sequence $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$.*

Proposition 1.2. *If $K \in \mathcal{C}(\mathcal{H})$, then K is bounded – i.e., $\mathcal{C}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. In addition, $\mathcal{C}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$.*

Proof. We leave this as an exercise for the reader. □

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of an operator K is finite dimensional, then we say that K is a finite-rank operator. For bounded set $B \in \mathcal{H}$, $K(B)$ is a bounded subset of a finite dimensional space and is therefore precompact. It follows that K is in $\mathcal{C}(\mathcal{H})$.

To describe K explicitly, let $\{\phi_k\}_{k=1}^{n}$ be a basis for $R(K)$. Then, $Kf = \sum_{k=1}^{n} a_k\phi_k$. We want to see how the a_k's depend on f. Consider $\langle Kf, \phi_j\rangle = \langle f, K^*\phi_j\rangle = \sum_{k=1}^{n} a_k\langle\phi_k, \phi_j\rangle$. Next let $\psi_j = K^*\phi_j$, so that $\langle f, K^*\phi_j\rangle = \langle f, \psi_j\rangle$. Because $\{\phi_k\}_{k=1}^{n}$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle\phi_k, \phi_j\rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j\rangle = \sum_{k=1}^{n} G_{j,k}a_k$. Doing so yields $a_k = \sum_{j=1}^{n}(G^{-1})_{k,j}\langle f, \psi_j\rangle$. The a_k's are obviously linear in f. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0, 1]$. A particularly important set of finite rank operators in $\mathcal{C}(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^{n} \phi_k(x)\overline{\psi_k(y)}$, where the functions involved are in L^2. The operator is then $Kf(x) = \int_0^1 k(x, y)f(y)dy$. In the example that we did for resolvents, the kernel was $k(x, y) = x^2y$, and the operator was $Ku(x) = \int_0^1 k(x, y)u(y)dy$. We will show that the Hilbert-Schmidt kernels also yield compact operators. This will follow as a corollary to our next theorem, which is especially important.

Theorem 1.3. *$\mathcal{C}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose that $\{K_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}(\mathcal{H})$ that converges to $K \in \mathcal{B}(\mathcal{H})$, in the operator norm. We want to show that K is compact. Assume the $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, with $\|v_k\| \le C$ for all k. Compactness will follow if we can prove that $\{Kv_k\}$ has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form $\{K_1 v_k\}$. Since K_1 is compact, we can select a subsequence $\{v_k^{(1)}\}$ such that $\{K_1 v_k^{(1)}\}$ is convergent. We may carry out the same procedure with $\{K_2 v_k^{(1)}\}$, selecting a subsequence of $\{K_2 v_k^{(1)}\}$ that is convergent. Call it $\{v_k^{(2)}\}$. Since this is a subsequence of $\{v_k^{(1)}\}$, $\{K_1 v_k^{(2)}\}$ is convergent. Continuing in this way, we construct subsequences $\{v_k^{(j)}\}$ for which $\{K_m v_k^{(j)}\}$ is convergent for all $1 \le m \le j$. Next, we let $\{u_j := v_j^{(j)}\}$, the "diagonal" sequence. This is a subsequence of all of the $\{v_k^{(j)}\}$'s. Consequently, for n fixed, $\{K_n u_j\}_{j=1}^{\infty}$ will be convergent. To finish up, we will use an "up, over, and around" argument. Note that for all ℓ, m,

$$\|Ku_\ell - Ku_m\| \le \|Ku_\ell - K_n u_\ell\| + \|K_n u_\ell - K_n u_m\| + \|K_n u_m - Ku_m\|$$

Since $\|Ku_\ell - K_n u_\ell\| \le \|K - K_n\|_{op}\|u_\ell\| \le 2C\|K - K_n\|_{op}$ and, similarly, $\|Ku_m - K_n u_m\| \le 2C\|K - K_n\|_{op}$, so we have $\|Ku_\ell - Ku_m\| \le 4C|K - K_n\|_{op} + \|K_n u_\ell - K_n u_m\|$. Let $\varepsilon > 0$. First choose N such that for $n \ge N$, $\|K - K_n\|_{op} < \varepsilon/(8C)$. Fix n. Because $\{K_n u_\ell\}$ is convergent, it is Cauchy. Choose N' so large that $\|K_n u_\ell - K_n u_m\| < \varepsilon/2$ for all $\ell, m \ge N'$. Putting these two together yields $\|Ku_\ell - K_n u_\ell\| \le \varepsilon$, provided $\ell, m \ge N'$. Thus $\{Ku_\ell\}$ is is Cauchy and therefore convergent. □

Corollary 1.4. *Hilbert-Schmidt operators are compact.*

Proof. Let $\mathcal{H} = L^2[0,1]$ and suppose $k(x,y) \in L^2(R)$, $R = [0,1] \times [0,1]$. The associated Hilbert-Schmidt operator is $Ku = \int_0^1 k(x,y)u(y)dy$. Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n. basis for $L^2[0,1]$. With a little work, one can show that $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}$ is an o.n. basis for $L^2(R)$. Also, from example 2 in the notes on Bounded Operators (11/7/13), we have that $\|K\|_{op} \le \|k\|_{L^2(R)}$. Expand $k(x,y)$ in the o.n. basis $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}$:

$$k(x,y) = \sum_{n,m=1}^{\infty} \alpha_{m,n}\phi_n(x)\phi_m(y),\ \alpha_{m,n} = \langle k(x,y), \phi_n(x)\phi_m(y)\rangle_{L^2(R)}$$

Next, let $k_N(x,y) = \sum_{n,m=1}^{N} \alpha_{m,n}\phi_n(x)\phi_m(y)$ and also K_N be the finite rank operator $K_N u(x) = \int_0^1 k_N(x,y)u(y)dy$. By Parseval's theorem, we have that $\|k-k_N\|^2_{L^2(R)} = \sum_{n,m=N+1}^{\infty} |\alpha_{m,n}|^2$ and by example 2 mentioned above, $\|K-K_N\|^2_{op} \le \|k-k_N\|^2_{L^2(R)}$, so

$$\|K-K_N\|^2_{op} \le \sum_{n,m=N+1}^{\infty} |\alpha_{m,n}|^2$$

Because the series on the right above converges to 0 as $N \to \infty$, we have $\lim_{N\to\infty} \|K-K_N\| = 0$. Thus K is the limit in $\mathcal{B}(L^2[0,1])$ of finite rank operators, which are compact. By the theorem above, K is also compact. □

We now turn to some of the algebraic properties of $\mathcal{C}(\mathcal{H})$.

Proposition 1.5. *Let $K \in \mathcal{C}(\mathcal{H})$ and let $L \in \mathcal{B}(\mathcal{H})$. Then both KL and LK are in $\mathcal{C}(\mathcal{H})$.*

Proof. Let $\{v_k\}$ be a bounded sequence in $\mathcal{H}$. Since L is bounded, the sequence $\{Lv_k\}$ is also bounded. Because K is compact, we may find a subsequence of $\{KLv_k\}$ that is convergent, so $KL \in \mathcal{C}(\mathcal{H})$. Next, again assuming $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, we may extract a convergent subsequence from $\{Kv_k\}$, which, with a slight abuse of notation, we will denote by $\{Kv_j\}$. Because L is bounded, it is also continuous. Thus $\{LKv_j\}$ is convergent. It follows that LK is compact. □

Proposition 1.6. *K is compact if and only if K^* is compact.*

Proof. Because K is compact, it is bounded and so is its adjoint K^*, in fact $\|K^*\|_{op} = \|K\|_{op}$. By Proposition 1.5, we thus have that KK^* is compact. It follows that if $\{v_n\}$ be a bounded sequence in $\mathcal{H}$, then we may extract a subsequence $\{v_j\}$ such that the sequence $\{KK^*v_j\}$ is convergent. This of course means that this sequence is also Cauchy. Note that

$$\langle KK^*(v_j-v_k), v_j-v_k\rangle = \langle K^*(v_j-v_k), K^*(v_j-v_k)\rangle = \|K^*(v_j-v_k)\|^2.$$

From and the fact that $\{v_j\}$ is bounded, we see that $\langle KK^*(v_j-v_k), v_j-v_k\rangle \le \|v_j-v_k\|\,\|KK^*(v_j-v_k)\| \le C\|KK^*(v_j-v_k)\|$. Thus,

$$\|K^*(v_j-v_k)\|^2 \le C\|KK^*(v_j-v_k)\|$$

Since $\{KK^*v_j\}$ is Cauchy, for every $\varepsilon > 0$, we can find N such that whenever $j,k \ge N$, $\|KK^*(v_j-v_k)\| < \varepsilon^2/C$. It follows that $\|K^*(v_j-v_k)\| < \varepsilon$, if $j,k \ge N$. This implies that $\{K^*v_j\}$ is Cauchy and therefore convergent. □

We want to put this in more algebraic language. Taking L to be compact in Proposition 1.5, we have that the product of two compact operators is compact. Since $\mathcal{C}(\mathcal{H})$ is already a subspace, this implies that it is an algebra. Moreover, by taking L to be just a bounded operator, we have that $\mathcal{C}(\mathcal{H})$ is a two-sided *ideal* in the algebra $\mathcal{B}(\mathcal{H})$. Since K being compact implies K^* is compact, $\mathcal{C}(\mathcal{H})$ is closed under the operation of taking adjoints; thus, $\mathcal{C}(\mathcal{H})$ is a $*$-ideal. Finally, including the result of Theorem 1.3, we have that $\mathcal{C}(\mathcal{H})$ is a closed under limits. We summarize these results as follows.

Theorem 1.7. *$\mathcal{C}(\mathcal{H})$ is a closed, two-sided, $*$-ideal in $\mathcal{B}(\mathcal{H})$.*