This problem assumes the axiom of choice.

Let k be a field, $\overline{k}$ its algebraic closure and G the group of automorphisms of $\overline{k}$ as a k algebra. The point of this problem is to prove that the maximal ideals of $k[x_1, \ldots, x_n]$ are in bijection with the G-orbits in k^n. When $k = \overline{k}$, this is the Nullstellansatz; you may use the Nullstellansatz in solving this problem. While this is not a logically crucial fact, many people become confused over what closed points are in an algebraically closed field, so I wanted to take this opportunity to clear it up.

(a) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \overline{k}^n$. Map $k[x_1, \ldots, x_n]$ to $\overline{k}$ by $x_i \mapsto \alpha_i$. Show that the kernel of this map is a maximal ideal. We will call this ideal $\mathfrak{m}(\alpha)$.

(b) Let α and β be two points of $\overline{k}^n$. Show that $\mathfrak{m}(\alpha) = \mathfrak{m}(\beta)$ if and only if there is some $g \in G$ taking α to β. (This is the part that uses AC for one direction.)

(c) Let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$. Show that $\overline{k}\mathfrak{m}$ is a proper ideal of $\overline{k}[x_1, \ldots, x_n]$ (not the whole ring).

(d) Show that every maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $\mathfrak{m}(\alpha)$ for some $\alpha \in \overline{k}^n$. (Hint: Part (c) is relevant, and you may use the Nullstellansatz.)