NOTES FOR JANUARY 11

GIWAN KIM

1. Differential Forms

Let X be a smooth manifold. A ***k-form*** on X is an object which, at each point $x \in X$, assigns a real value to k tangent vectors $v_1, \dots, v_k \in T_xX$. A k-form is linear and anti-symmetric in v_1, $\dots$, v_k, and varies smoothly with respect to x. Let $\Omega^k(X)$ denote the space of k-forms on X. If $x_1, \dots, x_n$ are local coordinates on X ($\dim X = n$) then a typical k-form looks like

$$\sum_I f_{i_1 \dots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad f_{i_1 \dots i_k}(x) \text{ is smooth}$$

and some typical identities are

$$\begin{aligned} f(x,y)\, dx \wedge dy &= -f(x,y)\, dy \wedge dx \\ &= f(x,y)\, dx \wedge (d(x+y)) \\ dx \wedge dx &= 0. \end{aligned}$$

1.1. **Wedge Product.** The ***wedge product*** is a map $\wedge : \Omega^k(X) \times \Omega^l(X) \longrightarrow \Omega^{k+l}(X)$ which is bilinear, associative, and anti-symmetric. Anti-symmetric means that given $\omega \in \Omega^k(X)$ and $\eta \in \Omega^l(X)$,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

Locally, the wedge product is defined by $\mathscr{C}^\infty$-linearly extending the map

$$(f(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},\, g(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}) \longmapsto f(x)g(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

In particular, anti-symmetry implies that $dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} = 0$ if any of the $i_1, \dots, i_k, j_1, \dots, j_l$ are repeated.

1.2. **Exterior Derivative.** The ***exterior derivative*** is a linear map $d : \Omega^k(X) \longrightarrow \Omega^{k+1}(X)$ for $k \geqslant 0$. For f a 0-form, that is to say, a smooth function, we have

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

In general, d is given by

$$d\left(\sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) = \sum_I df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

which can be checked to satisfy $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta$. Given a smooth function f and $v \in T_xX$,

$$(df)(x)(v) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}$$

where $\gamma : (-\epsilon, \epsilon) \longrightarrow X$ is a path such that $\gamma(0) = x$ and $\gamma'(0) = v$.

1.3. **Stokes' Theorem.** There is a coordinate free formula for the exterior derivative: Let $\omega \in \Omega^k(X)$ and $V_1, \dots, V_{k+1}$ be smooth vector fields on X. Then

$$\begin{aligned} d\omega(V_1, \dots, V_{k+1}) = &\sum_{1 \leqslant i \leqslant k+1} (-1)^{i-1} V_i(\omega(V_1, \dots, \widehat{V_i}, \dots, V_{k+1})) \\ &+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \omega([V_i, V_j], V_1, \dots, \widehat{V_i}, \dots, \widehat{V_j}, \dots, V_{k+1}), \end{aligned}$$

where the hats indicate omitted arguments. Suppose we have $k+1$ commuting flows $V_1, \ldots, V_{k+1}$ ($[V_i, V_j] = 0$) and let $\omega(0, \ldots, t, \ldots, 0) := \omega(\phi_i(t))$ where $\phi_i : X \times (-\epsilon, \epsilon) \longrightarrow X$ is the flow adapted to V_i. Then, by the above equation

$$(d\omega)(V_1, \ldots, V_{k+1}) = \frac{\partial}{\partial t}\Big|_{t=0} \omega(t, 0, \ldots, 0)(V_2, \ldots, V_{k+1}) - \frac{\partial}{\partial t}\Big|_{t=0} \omega(0, t, 0, \ldots, 0)(V_1, V_3, \ldots, V_{k+1}) +$$

$$\cdots + (-1)^k \frac{\partial}{\partial t}\Big|_{t=0} \omega(0, \ldots, 0, t)(V_1, \ldots, V_k). \tag{1}$$

We can integrate a (compactly supported) k-form on a k-dimensional (orientable) submanifold of X by the usual Riemann integral $\sum_{\text{mesh}} \omega(x)(V_1, \ldots, V_k)$, where V_i are commuting vector fields locally on the submanifold, and taking the limit as the size of mesh approaches zero, then add the integrals defined locally together by partition of unity.

Consider a $(k+1)$–box with each side length t and coordinates given by flows adapted to commuting vector fields, $V_1, \ldots, V_{k+1}$. Notice that

$$\frac{\partial}{\partial t}\Big|_{t=0} \omega(t, 0, \ldots, 0)(V_2, \ldots, V_{k+1}) = \lim_{t \to \infty} \frac{1}{t} \left(\frac{1}{t^k} \int_{\text{Face 1}} \omega - \frac{1}{t^k} \int_{\text{Face 2}} \omega \right) \tag{2}$$

and similarly for other terms (signs workout!). Observe that

$$\int_{\text{Box}} d\omega \approx t^{k+1} \left(\frac{\partial}{\partial t}\Big|_{t=0} \omega(t, 0, \ldots, 0)(V_2, \ldots, V_{k+1}) + \cdots + (-1)^k \frac{\partial}{\partial t}\Big|_{t=0} \omega(0, \ldots, 0, t)(V_1, \ldots, V_k) \right)$$

for $t \ll 0$ by (1). The right-hand side is $t^{k+1} \int_{\partial\text{Box}} \omega + o(t^{k+1})$ by (2) which shows that for $t \ll 0$,

$$\int_{\partial\text{Box}} \omega \approx \int_{\text{Box}} d\omega.$$

As a matter of fact, more is true:

Theorem 1. *(Stokes' Theorem) If B is a k-dimensional oriented submanifold of X with boundary, then for compactly supported k-form ω*

$$\int_{\partial B} \omega = \int_B d\omega.$$

1.4. **Pullback.** Let $F : X \longrightarrow Y$ be a smooth map and ω be a k-form on Y. We define $F^*\omega$ a k-form on X by

$$(F^*\omega)(x)(v_1, \ldots, v_k) = \omega(F(x))(F_*(v_1), \ldots, F_*(v_k))$$

where $v_1, \ldots, v_k \in T_xX$ and $F_* : T_xX \longrightarrow T_{F(x)}Y$ is a linear map. Some properties of F^* are

(1) $F^*(\omega + \eta) = F^*\omega + F^*\eta$.
(2) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$.
(3) $F^*(d\omega) = d(F^*\omega)$.
(4) If F restricts to a diffeomorphism between B, a k-dimensional submanifold X and C, a k-dimensional submanifold of Y, then

$$\int_B F^*\omega = \int_C \omega.$$

2. Poincaré Lemma

Let U be a contractible open subset of $\mathbb{R}^n$. Then,

$$\Omega^{k-1}(U) \xrightarrow{d} \Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U)$$

is exact for $0 < k \leqslant n$.

The point is we want to define an operator s which takes a closed k-form ω and gives a $(k-1)$-form $s\omega$ such that $d(s\omega) = \omega$. As a matter of fact, we will define $s : \Omega^k(U) \longrightarrow \Omega^{k-1}(U)$ such that for any ω, we have $ds\omega + sd\omega = \omega$. Hence, if $d\omega = 0$ then $sd\omega = 0$ (s will be linear) so $ds\omega = \omega$. That is to say, s is a chain homotopy between identity map and zero map:

$$\begin{array}{ccccc} \Omega^{k-1}(U) & \longrightarrow & \Omega^k(U) & \xrightarrow{d} & \Omega^{k+1}(U) \\ & \swarrow^{s} & \downarrow{\scriptstyle \mathrm{Id}} & \swarrow^{s} & \\ \Omega^{k-1}(U) & \xrightarrow{d} & \Omega^k(U) & \longrightarrow & \Omega^{k+1}(U) \end{array}$$

where $ds + sd = \mathrm{Id} - 0$.

Since U is contractible, there exists a smooth map $\rho : U \times [0,1] \longrightarrow U$ such that $\rho|_{U\times\{0\}}$ is the constant map to a point $u_0 \in U$ and $\rho|_{U\times\{1\}} = \mathrm{Id}$. Suppose $\omega \in \Omega^{k-1}(U)$ such that

$$\rho^*\omega = \sum_I f_{_I}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum_J g_{_J}\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}.$$

We define

$$s\omega := \sum_J \left(\int_0^1 g_{_J}\, dt \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}.$$

Notice that

$$\rho^*(d\omega) = d(\rho^*\omega) = \sum_{I,l} \frac{\partial f_{_I}}{\partial x_l}\, dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum_I \frac{\partial f_{_I}}{\partial t}\, dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ \sum_{J,l} \frac{\partial g_{_J}}{\partial x_l}\, dx_l \wedge dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

which implies that

$$s(d\omega) = \sum_I \left(\int_0^1 \frac{\partial f_I}{\partial t}\, dt \right) dx_I - \sum_{J,l} \left(\int_0^1 \frac{\partial g_J}{\partial x_l}\, dt \right) dx_l \wedge dx_J.$$

Also

$$d(s\omega) = \sum_{J,l} \frac{\partial}{\partial x_l} \left(\int_0^1 g_{_J}\, dt \right) dx_l \wedge dx_{_J} = \sum_{J,l} \left(\int_0^1 \frac{\partial g_{_J}}{\partial x_l}\, dt \right) dx_l \wedge dx_{_J},$$

hence

$$d(s\omega) + s(d\omega) = \sum_I \left(\int_0^1 \frac{\partial f_{_I}}{\partial t}\, dt \right) dx_{_I} = \sum_I \Big(f_{_I}(x,1)\, dx_{_I} - f_{_I}(x,0)\, dx_{_I} \Big)$$
$$= \rho(x,1)^*\omega - \rho(x,0)^*\omega = \omega$$

since $\rho(x,1) = \mathrm{Id}_U$ and $\rho(x,0) =$ constant map.

Second proof: Suppose $U = (a_1, b_1) \times \cdots \times (a_n, b_n)$. and let ω be a closed k-form. We induct on the largest p such that dx_p appears in ω. If no such p appears then $\omega = 0$ and $\omega = d \cdot 0$ so ω is trivially exact. In general, suppose

$$\omega = \sum_I f_{_I}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_I f_{_I}\, dx_{_I}.$$

For q larger than p, the coefficient of $dx_q \wedge dx_{_I}$ in $d\omega$ is $\frac{\partial f_I}{\partial x_q}$ since q does not appear in I. Therefore, $\frac{\partial f_I}{\partial x_q} = 0$ ($d\omega = 0$). As U is connected, $f_{_I}$ is constant with respect to x_q where $q > p$.

Set

$$\alpha = \sum_I \left(\int_{a_p}^{b_p} f_{_I}(x_1, \ldots, x_{p-1}, t, x_{p+1}, \ldots, x_n)\, dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}.$$

Then,

$$d\alpha = \sum_{p \in I} f_{_I} dx_{_I} + \sum_{J \subseteq \{1,\ldots,p-1\}} g_{_J} dx_{_J}$$

so $\omega - d\alpha$ is a sum of $dx_{_K}$ with $K \subseteq \{1, \ldots, p-1\}$. (We have used that $f_{_I}$ is constant with respect to x_q to see that the integral is constant with respect to x_q for $q > p$.) By inductive hypothesis, $\omega - d\alpha = d\beta$ since $d(\omega - d\alpha) = d\omega = 0$. Hence, $\omega = d(\alpha + \beta)$.