MATH 632 NOTES: LINE BUNDLES AND DIVISORS

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Last time, we talked about holomorphic line bundles. We have a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

Let X be a compact Kähler manifold, and let X be a divisor, i.e. an integer linear combination of complex submanifolds of codimension one. Recall that $[D] \in H_{2n-2}(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. For any $\eta \in \Omega^{n,n-2}(X)$, we have $\eta|_D = 0$, so $\int_D$ is 0 on $H^{n,n-2}(X)$. Thus $[D]$ maps to 0 in $H^{0,2}(X) = H^{n,n-2}(X)^*$ and $[D] \in \mathrm{NS}(X)$.

Here are some natural questions:

- What line bundles can we build from D?
- We know line bundles give us classes in $H^{1,1}(X)$ (or more generally $H^1(\mathcal{Z}^1)$ for non-Kähler manifolds), but do they give us closed (1,1)-forms?

Line bundles from divisors

Given any complex manifold X and a hypersurface D, we define $\mathcal{O}(-D) \subset \mathcal{O}$ by

$$\mathcal{O}(-D)(U) = \{f \in \mathcal{O}(U) \mid f|_D = 0\}.$$

Recall that by a version of the implicit function theorem, there is an open cover of X such that D is principal on each chart of the cover. More precisely, at each point of D there exist local coordinates $z_1, \cdots, z_n$ such that $D = \{z_n = 0\}$. In that chart,

$$\mathcal{O}(-D)(U) = z_n \cdot \mathcal{O}(U) \cong \mathcal{O}(U),$$

so $\mathcal{O}(-D)$ is locally free.

On the other hand, we let $\mathcal{O}(D)(U)$ consist of meromorphic functions on U which have at most a simple pole at D but are well-defined elsewhere. So locally

$$\mathcal{O}(D)(U) = z_n^{-1} \cdot \mathcal{O}(U) \cong \mathcal{O}(U).$$

In general, for $a_i \in \mathbb{Z}$ we define

$$\mathcal{O}(\sum a_i D_i) = \bigotimes \mathcal{O}(D_i)^{\otimes a_i},$$

where for $a_i < 0$ we mean $\mathcal{O}(D_i)^{\otimes a_i} = \mathcal{O}(-D_i)^{\otimes -a_i}$.

Warning: Let g be a function in $\mathcal{O}(U)$ which vanishes to order k along D. Then, as a section of $\mathcal{O}(-D)$, g vanishes to order $k-1$. In general, if $g \in \mathcal{O}(\sum a_i D_i)(U)$ has a zero of order b_i at D_i, then as a section g has a zero of order $b_i + a_i$.

In gluing data: choose some U_i such that D is cut out by z_i. Then $g_{j \leftarrow i} = z_j z_i^{-1}$ for $\mathcal{O}(-D)$, or $z_i z_j^{-1}$ for $\mathcal{O}(D)$, which defines a Čech cocycle for $H^1(X, \mathcal{O}^*, U_\bullet)$.

In fact, $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ sends $\mathcal{O}(\sum a_i D_i) \mapsto \sum a_i [D_i]$, so our construction does what we want.

Note: there are Kähler manifolds with line bundles not of this form. A given line bundle L has the form $\mathcal{O}(\sum a_i D_i)$ if and only if L has a nonzero meromorphic section.

$(1,1)$ forms from connections

In general, given a complex manifold X and a holomorphic line bundle, can we get a specific closed (1,1)-form? Yes, but first we need a positive definite Hermitian form on L.

Recall that because L is holomorphic, we have a connection $\overline{D} : L \to L \otimes \Omega^{0,1}$. Also, there is a unique Chern connection $\nabla = D + \overline{D}$ which preserves the norm on L. Then

$$\nabla^2 = (D + \overline{D})(D + \overline{D}) = D^2 + D\overline{D} + \overline{D}D + \overline{D}^2 = D\overline{D} + \overline{D}D,$$

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and $(D\overline{D}+\overline{D}D)\sigma = \Theta\sigma$ for Θ a closed (1,1)-form. In the rest of these notes, we verify that $\frac{1}{2\pi i}\Theta$ represents the correct class $H^1(\mathcal{Z}^1)$, or $H^{1,1}(X)$ if X is Kähler.

Take an open cover U_i where L is trivial, and let σ_i be a nonzero holomorphic section on U_i. Put $h_i = \sqrt{\langle \sigma_i, \sigma_i \rangle} = |\sigma_i|$, so $\sigma_i = h_i e_i$ where e_i is a smooth section of norm 1. Then

$$\overline{D}e_i = \overline{D}(h_i^{-1}\sigma_i) = \overline{\partial}h_i^{-1}\sigma_i + h_i^{-1}\overline{D}\sigma_i = \overline{\partial}h_i^{-1}\sigma_i = (-h_i^{-2}\overline{\partial}h_i)(h_i e_i) = -\frac{\overline{\partial}h_i}{h_i}e_i.$$

So we see that, in the e_i trivialization, we have $\overline{D} = \overline{\partial} - \frac{\overline{\partial}h_i}{h_i}$. So

$$\nabla = D + \overline{D} = d - \frac{\overline{\partial}h_i}{h_i} + \overline{\frac{\overline{\partial}h_i}{h_i}} = d - \frac{\overline{\partial}h_i}{h_i} + \frac{\partial h_i}{h_i},$$

where $\overline{\frac{\overline{\partial}h_i}{h_i}} = \frac{\partial h_i}{h_i}$ because h_i is real-valued.

In general, if we have a line bundle with a connection ∇ which is $d+\alpha$ in local coordinates, then the curvature is

$$\nabla^2(f) = d\,(df + f\alpha) + \alpha \wedge (df + f\alpha) = df \wedge \alpha + f \wedge d\alpha + \alpha df = (d\alpha)f.$$

So the curvature is $d\alpha$. See Problem Set 6, Problem 1. (I also thought I talked about this on March 15, but it isn't in the scribed notes.)

In our case, we get that the curvature of ∇ is

$$\begin{aligned} \Theta &= d\left(\tfrac{\partial h_i}{h_i} - \tfrac{\overline{\partial}h_i}{h_i}\right) \\ &= (\partial + \overline{\partial})(\partial \log h_i - \overline{\partial}\log h_i) \\ &= \overline{\partial}\partial \log h_i^2 && (\partial \text{ and } \overline{\partial} \text{ anticommute}) \\ &= \overline{\partial}\partial \log \langle \sigma_i, \sigma_i \rangle. \end{aligned}$$

Remark: This formula must be independent of the choice of σ_i, since the curvature is determined by the connection and the connection is determined by the holomorphic structure and by the metric. It is a good exercise to see this directly. Any other holomorphic section τ_i would be of the form $g\sigma_i$, for $g \in \mathcal{O}^*(U_i)$. We have

$$\overline{\partial}\partial \log\langle \tau_i, \tau_i \rangle = \overline{\partial}\partial \log \langle \sigma_i, \sigma_i \rangle + \overline{\partial}\partial \log g + \overline{\partial}\partial \log \overline{g}.$$

The latter two terms are zero because $\log g$ is holomorphic and $\log \overline{g}$ is anti-holomorphic. This uniqueness implies that the $\overline{\partial}\partial \log\langle \sigma_i, \sigma_i \rangle$ agree on overlaps, hence glue to a global (1,1)-form.

We now resume our verification that $\Theta/(2\pi i)$ is the desired closed $(1,1)$-form. Recall that our $(1,1)$ form arises from the map $\mathcal{O}^* \to \mathcal{Z}^1$ which sends $f \mapsto \frac{1}{2\pi i}\frac{\partial f}{f}$. Put $g_{j \leftarrow i} = \sigma_j \sigma_i^{-1}$, so that the cocycle we get in $H^1(\mathcal{Z}^1)$ is $U_i \cap U_j \mapsto \frac{\partial \sigma_j}{\sigma_j} - \frac{\partial \sigma_i}{\sigma_i}$. We need to check that this corresponds to $\overline{\partial}\partial \log\langle \sigma_i, \sigma_i \rangle$ as a (1,1)-form.

Now look at the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{Z}^1 \longrightarrow \{\partial - \text{closed (1,0)-forms}\} \xrightarrow{\overline{\partial}} \{d - \text{closed (1,1)-forms}\} \longrightarrow 0,$$

which induces a map on cohomology:

$$H^0(d - \text{closed (1,1)-forms}) \longrightarrow H^1(\mathcal{Z}^1).$$

We need to show that this boundary map takes $\overline{\partial}\partial \log\langle \sigma_i, \sigma_i \rangle$ to the Čech cocyle $U_i \cap U_j \mapsto \frac{\partial \sigma_j}{\sigma_j} - \frac{\partial \sigma_i}{\sigma_i}$.

Recall how to compute the Čech boundary map: Lift $\overline{\partial}\partial \log\langle \sigma_i, \sigma_i \rangle$ to $U_i \mapsto \partial \log\langle \sigma_i, \sigma_i \rangle$, which is a Čech cochain for $\{\partial - \text{closed (1,0)-forms}\}$. Then take the difference on overlaps to get

$$\begin{aligned} U_i \cap U_j \mapsto \partial \log\langle \sigma_i, \sigma_i \rangle - \partial \log \langle \sigma_j, \sigma_j \rangle &= \partial \log \left|\tfrac{\sigma_i}{\sigma_j}\right|^2 = \partial \log \tfrac{\sigma_i}{\sigma_j} + \partial \log \overline{\tfrac{\sigma_i}{\sigma_j}} \\ &= \partial \log \tfrac{\sigma_i}{\sigma_j} = \tfrac{\partial \sigma_i}{\sigma_i} - \tfrac{\partial \sigma_j}{\sigma_j}. \qquad \square \end{aligned}$$

Here the equality at the line break is because ∂ of an anti-holomorphic function is 0.