NOTES FOR APRIL 5TH

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1. The map $H^k(X,\mathbb{Z}) \to H^k_{DR}(X,\mathbb{C})$

Given a geometric cocycle, how does it decompose under Hodge decomposition?

The bridge between the two is deRham cohomology through the following sequence of maps:

$$H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{Z}) \otimes \mathbb{C} \cong H^k(X,\mathbb{C}) \to H^k_{DR}(X,\mathbb{C}).$$

The matrix form of this map is more complicated and we first study the following easier map on a smooth n-dimensional manifold.

$$H_k(X,\mathbb{Z}) \to Hom(H^k_{DR}(X),\mathbb{C})$$

$$\sigma \mapsto (\omega \mapsto \int_\sigma \omega)$$

On Problem set 3, we did some non-trivial chasing through diagrams and checked this for k = 1, 2. We know from algebraic topology that $H_k(X,\mathbb{C})$ is dual to $H^k(X,\mathbb{C})$. $Hom(H^k_{DR}(X,\mathbb{C}))$ is dual to $H^k_{DR}(X)$ by definition. Therefore

$$H^k_{DR}(X) \stackrel{\cong}{\to} H^k(X,\mathbb{C})$$

is the matrix transpose of the above isomorphism, ie. mapping ω to $(\sigma \mapsto \int_\sigma \omega)$ The messy one is to invert this matrix to get the isomorphism from $H^k(X,\mathbb{C})$ to $H^k_{DR}(X)$. The clever way is as follows. Assume X is compact, oriented of dimension m. First, we have Poincare duality, as proved in (for example) Hatcher's book:

$$H^k(X,\mathbb{Z}) \cong H_{m-k}(X,\mathbb{Z})$$

by capping with the fundamental class $[X]$. And we have Poincare duality as proved in this class: $H^k_{DR}(X) \to H^{m-k}_{DR}(X)^*$ by $\alpha \mapsto (\beta \mapsto \int_X \alpha \wedge \beta)$. These fit together into the right hand of the following diagram:

$$\begin{array}{ccccc}
H^k_{DR}(X) & \xrightarrow{\cong} & H^{m-k}_{DR}(X)^* & \longrightarrow & H^{p,q}(X) \\
\uparrow & & \uparrow & & \downarrow \\
H^k(X,\mathbb{Z}) & \xrightarrow{(\ )^{-1}} & H^k_{DR}(X) & & \\
\downarrow \cong & & \downarrow \cong & & \downarrow \\
H_{m-k}(X,\mathbb{Z}) & \longrightarrow & H^{m-k}_{DR}(X) & \longrightarrow & H^{n-p,n-q}(X)^*
\end{array}$$

This gives us a recipe to go from $H^k(X)$ to $H^k_{DR}(X)$, going down, right, up. That is to say, if $\alpha \in H^k(X,\mathbb{Z})$ maps to $\eta \in H^k_{DR}(X)$, then, for any $\theta \in H^{m-k}_{DR}(X)$, we have

$$\int_X \eta \wedge \theta = \int_{\alpha \cap [X]} \theta.$$

Now, suppose we want to understand how this all ties into Hodge decomposition. Suppose that X is compact Kähler, of complex dimension n. We then need the right hand side of the above diagram. The projection $H^k_{DR} \to H^{p,q}(X)$ is dual to the inclusion $H^{n-p,n-q}(X) \to H^{2n-k}$. In other words, if α maps to η in $H^{p,q}(X)$ then, for any $(n-p, n-q)$-form η, we have the above equality.

2. Line Bundles

Line bundles are rank one holomorphic bundles. They are usually given either with glueing data or by a divisor.

Let's start with glueing data: let L be a holomorphic line bundle over X, a complex manifold. Then the glueing data of L is $\{U_i\}$ an open cover on which L is trivial and sections σ_i on U_i. On $U_i \cap U_j$, $\sigma_i = g_{i \leftarrow j}\sigma_j$ for $g_{i\leftarrow j} \in \mathcal{O}^*(U_i \cap U_j)$. Here $\mathcal{O}^*$ is the sheaf of nowhere zero holomorphic functions.

The g's obey the Cech cocycle condition, so they can be seen as elements of $H^1(X, \mathcal{O}^*)$. $H^1(X, \mathcal{O}^*)$ is actually isomorphic to the isomorphism class of holomorphic line bundles on X whose group operation is given by tensor product.

We have a short exact sequence of sheaves:

$$0 \to L\mathbb{C}_{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

From this, we get long exact sequence of cohomologies:

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

We're interested in the middle term, so define

$$\mathrm{Pic}^0(X) := \mathrm{Coker}(H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}))$$

$$NS(X) := \mathrm{Ker}(H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}))$$

Note that $NS(X)$ is a discrete group. $\mathrm{Pic}^0(X)$ is a quotient of vector space. It's connected and divisible. If $H^1(X, \mathbb{Z})$ is discrete in $H^1(X, \mathcal{O})$, which is (for example) the case when X is compact Kähler, then $\mathrm{Pic}^0(X)$ is topologically a torus. (An abelian group G is divisible if, for any $g \in G$ and any nonzero integer n, there is an $h \in G$ st. $nh = g$.)

By the above definitions, we have

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X.

In particular, if $H^1(X, \mathcal{O}) = 0$ and $H^2(X, \mathbb{Z}) = 0$ then $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*) = (e)$.

Let's look at the discrete part of the picture. We defined $NS(X)$ as $\mathrm{Ker}(H^2(\mathbb{Z}) \to H^2(\mathcal{O}))$. If X is compact Kahler, then

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

$NS(X)$ is those integer classes not in $H^2(\mathcal{O}) = H^{0,2}$. Complex conjugation fixes $H^{1,1}$, exchanges $H^{2,0}$ with $H^{0,2}$ and fixes $H^2(X, \mathbb{Z})$. So $NS(X)$ consists of classes in $H^2(X, \mathbb{Z})$ that map to $H^{1,1}(X)$.

Note: $\mathbb{Z}^6 \to \mathbb{C}$ can be an injection. In this way, we can have compact Kahler manifolds with trivial NS(X).

Compatibility of various maps

Hunter Brooks raised a question. In the above discussion, we got maps from $H^2(X,\mathbb{Z})$ to $H^2(X,\mathcal{O})$ in 2 ways, one is through the exponential sequence, the other by tensoring the cohomology with $\mathbb{C}$, then projection by Hodge decomposition. Hunter's question was whether these two maps are the same.

The inclusion of sheaf $\mathbb{Z}$ in $\mathcal{O}$ can be factored as $\mathbb{Z} \stackrel{2\pi i}{\to} \mathbb{C} \to \mathcal{O}$. Our first map is the map on H^2 derived from the composite of these maps. By functoriality of cohomology, we can break this up into $H^2(\mathbb{Z}) \to H^2(\mathbb{C}) \to H^2(\mathcal{O})$. The first map is precisely the "tensor with $\mathbb{C}$" part of our second map. So the thing that remains to see is that the projection $H^2(X,\mathbb{C}) \to H^2(X,\mathcal{O})$ coming from the Hodge decomposition is the one coming from the map of sheaves $\mathbb{C} \to \mathcal{O}$. This is indeed true; it is perhaps easiest to think about this in terms of harmonic forms.

Even if X is not compact Kahler, we have:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow {\scriptstyle 2\pi i} & & & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O} & \xrightarrow{\partial} & Z^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle \exp} & & \downarrow {\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathcal{O}^* & \xrightarrow{g \mapsto \partial g/g} & Z^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where Z^1 is closed $(1,0)$-forms. The rows and columns are exact and the diagram commutes.

There are two ways to go from $H^1(\mathcal{O}^*)$ to $H^2(X,\mathbb{C})$ that we can observe from the previous diagram. One way is $H^1(\mathcal{O}^*) \to H^1(Z^1) = H^1(Z^1) \to H^2(\mathbb{C})$, using the horizontal exact sequences. The other is $H^1(\mathcal{O}^*) \to H^2(\mathbb{Z}) = H^2(\mathbb{Z}) \to H^2(\mathbb{C})$, using the vertical exact sequences.These maps are equal, which is a general homological fact.

$\mathrm{NS}(X)$ is the image of $H^1(\mathcal{O}^*)$ in $H^2(\mathbb{Z})$; we thus see that we can describe $\mathrm{NS}(X)$ as the subgroup of $H^2(X,\mathbb{Z})$ consisting of those classes that have image in $H^2(X)$ representable by a closed $(1,1)$-form.

Divisors and line bundles

If X is compact Kähler then, as discussed at the beginning of lecture, $NS(X)$ is the subset of $H^2(X,\mathbb{Z}) = H_{2n-2}(X,\mathbb{Z})$ corresponding to codimension 2 cycles which integrate to 0 against all closed $(n, n-2)$-forms.

Similarly,

$$\mathrm{Pic}^0(X) \cong H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) \cong H^{0,1}(X)/H^1(X,\mathbb{Z}) \cong$$
$$H^{n,n-1}(X)^*/\left(\text{functionals of the form } \int_\sigma \text{ for } \sigma \in H_{2n-1}(X,\mathbb{Z})\right).$$

In particular, if $D \subset X$ is a complex submanifold of dimension $n-1$, then $\int_d \omega = 0$ for any $\omega \in H^{n,n-2}(X)$ as $\omega|_D = 0$ So there should be some line bundle whose Neron-Severi class is D (which will prove two classes later).

We can also give a geometric way of thinking of the description of $\mathrm{Pic}^0(X)$ as $H^{n,n-1}(X)^*$ modulo integration over closed $2n-1$ cycles. Specifically, suppose $D \subset X$ has class 0 in $H_{2n-2}(X)$, so $D = \partial M$, where M is a real $(2n-1)$-fold. Then we get a map $\int_M : H^{n,n-1}(X) \to \mathbb{C}$, so M gives a point in $H^{n,n-1}(X)^*$.

If we choose a different M' with $D = \partial M'$, then $\partial(M - M') = 0$ so $M - M' \in H_{2n-1}(X, \mathbb{Z})$. Thus we have a well defined map $[\mathcal{O}(D)] \mapsto [\int_M]$ identifying Pic^0 with $H^{n,n-1}(X)^*$/integration over $H_{2n-1}(X, \mathbb{Z})$