PROBLEM SET 12
DUE APRIL 19, 2011

This problem set is probably a bit too long. Choose the four problems you'd most like to write up.

1 The point of this problem is to work out how connections work on tensor products. Let E and F be two vector bundles over a smooth manifold X, and let ∇_E and ∇_F be connections on them.

(1) Define $\nabla_{E \otimes F} : E \otimes F \to E \otimes F \otimes \Omega^1_X$ by $\nabla_{E\otimes F}(\sigma \otimes \tau) = \nabla_E(\sigma) \otimes \tau + \sigma \otimes \nabla_F(\tau)$. Show that, for any $f \in C^\infty X$, this formula gives the same result on $(f\sigma) \otimes \tau$ and on $\sigma \otimes (f\tau)$, so it is a well defined connection.

(2) Let $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ be bilinear forms on E and on F. Define a bilinear form $\langle\ ,\ \rangle_{E\otimes F}$ by
$$\langle \sigma_1 \otimes \tau_1, \sigma_2 \otimes \tau_2 \rangle_{E \otimes F} = \langle \sigma_1, \sigma_2 \rangle \langle \tau_1, \tau_2 \rangle$$
and extending linearly. Show that, if ∇_E and ∇_F preserve $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$, then $\nabla_{E\otimes F}$ preserves $\langle\ ,\ \rangle_{E\otimes F}$. (See Problem Set 6, Problem 2 for the notion of a connection preserving a bilinear form.)

(3) Let X be a complex manifold; let E and F be holomorphic vector bundles; let $\overline{D}_E$ and $\overline{D}_F$ be the corresponding $(0,1)$-connections. Define $\overline{D}_{E\otimes F}$ analogously to how we defined $\nabla_{E\otimes F}$ above. Show that, if σ and τ are holomorphic sections of E and F, then $\overline{D}_{E\otimes F}(\sigma \otimes \tau) = 0$. This checks that $\overline{D}_{E\otimes F}$ is the $(0,1)$ connection for the holomorphic structure on $E \otimes F$.

2. Let X be a compact Kähler manifold. Let L be a holomorphic line bundle on X, equipped with an inner product $\langle\ ,\ \rangle$. Let $\nabla = D + \overline{D}$ be the corresponding connection, and let Θ be the curvature: The closed $(1,1)$-form such that $\nabla^2 \sigma = \sigma\Theta$ for any section σ.

(1) Check that Θ is purely imaginary, meaning that it assigns an imaginary number to any pair of real tangent vectors. (Look at the formula for Θ from the April 7 lecture.)

(2) Replace the metric $\langle\ ,\ \rangle$ by $e^\beta \langle\ ,\ \rangle$ where β is some real valued function. Let Θ' be the corresponding curvature. What is the relation between Θ, Θ' and β?

(3) Suppose that we are given a closed $(1,1)$-form Θ' such that Θ and Θ' represent the same class in $H^2(X)$. Show that there is a complex valued function β such that Θ, Θ' and β obey the relation you found in the previous part.

(4) Suppose that Θ' is as in the previous problem, and is purely imaginary. Let β be the function found in the previous part, such that (Θ, Θ', β) obeys the relation from part (2). Show that $(\Theta, \Theta', \mathrm{Re}(\beta))$ also obeys the relation from part (2).

Remark: We have now shown a lemma we will need in class: If X is compact Kähler, L is a holomorphic line bundle, and Θ' is a purely imaginary closed $(1,1)$-form representing the cohomology class of L, then there is a metric on L such that the connection has curvature Θ'.

3. The goal of this problem is to compute the various Hodge groups for the projective plane $\mathbb{P}^2$. We write $(x_1 : x_2 : x_3)$ for the homogenous coordinates on $\mathbb{P}^2$.

Let U_1, U_2, U_3 be the open sets on which x_1, x_2 and x_3 are nonzero. So $U_3 \cong \mathbb{C}^2$, with the isomorphism given by the coordinates x_1/x_3 and x_2/x_3, and likewise for the other two charts.

(1) Write down the Čech complex for $\mathcal{O}$ with respect to the cover $U_\bullet$. (For example, $\mathcal{O}(U_3)$ is everywhere convergent power series in x_1/x_3 and x_2/x_3.) Verify that $H^0(\mathcal{O}) \cong \mathbb{C}$ and $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$.

(2) Let $\eta_{ij} = d(x_i/x_j)$, this is a meromorphic $(1,0)$ form. For (i,j) equal to $(1,2)$, $(2,1)$, $(3,1)$ and $(1,3)$, find meromorphic functions a_{ij} and b_{ij} such that $\eta_{ij} = a_{ij}\eta_{13} + b_{ij}\eta_{23}$.
(3) Write down the Čech complex for $\mathcal{H}^1$ with respect to the cover $U_\bullet$. Describe each term as "the space of forms $a\eta_{13} + b\eta_{23}$ where a and b are of the form ...".
(4) Write down the Čech complex for $\mathcal{H}^2$ with respect to the cover $U_\bullet$. Describe each term as "the space of forms $a(\eta_{13} \wedge \eta_{23})$ where a is of the form ...".
(5) Compute that $H^0(\mathcal{H}^2) = 0$, $H^1(\mathcal{H}^2) = 0$ and $H^2(\mathcal{H}^2) \cong \mathbb{C}$.
(6) Compute that $H^0(\mathcal{H}^1) = 0$, $H^1(\mathcal{H}^1) \cong \mathbb{C}$ and $H^2(\mathcal{H}^1) = 0$.

Remark: The ordering of (3)-(6) are meant to be in order of difficulty.

4. (The Hilbert polynomial) Let X be a smooth d-dimensional complex submanifold of $\mathbb{P}^M$. Let ω be the restriction of the Fubini-Study form from $\mathbb{P}^M$. Bertini's theorem states that there is a hyperplane H such that $X \cap H$ is smooth; assume this in this problem.

(1) Let H be a hyperplane such that $X \cap H$ is smooth. On X, show that we have a short exact sequence of sheaves:
$$0 \to \mathcal{O}(N-1)|_X \to \mathcal{O}(N)|_X \to \mathcal{O}(N)|_{X\cap H} \to 0.$$
(2) Show that, for N sufficiently large, we have $\dim H^0(X, \mathcal{O}(N)) - \dim H^0(X, \mathcal{O}(N-1)) = \dim H^0(X \cap H, \mathcal{O}(N))$.
(3) Show that there is a polynomial h_X dependent on X such that, for N sufficiently large, we have $\dim H^0(X, \mathcal{O}(N)) = h_x(N)$. (Hint: Induction on d.)
(4) Show that h_X has degree d.
(5) Show that the leading term of h_X is $\int_X \omega^d N^d/d!$.

5. Let X be a compact[1] complex manifold and D a smooth hypersurface. In the past two lectures, we have seen that (1) D gives rise to a line bundle $\mathcal{O}(-D)$ and (2) give a metric on $\mathcal{O}(D)$, we get a closed $(1,1)$ form on X. Choose U to be an open set containing D. Our goal in this problem is to show that we can take ω to be 0 on $X \setminus U$.

The first part is a partition of unity argument.

(1) Show that there is a finite open cover $\bigcup V_i \cup W$ of X such that (a) each V_i is in U, (b) In each V_i, the divisor D is cut out by some holomorphic function z_i and (c) the closure $\overline{W}$ of W is disjoint from D.
(2) Let $1 = \sum \rho_i + \sigma$ be a partition of unity subject to the cover $\bigcup V_i \cup W$ (So ρ_i is a function on V_i, and σ is on W. Recall that the functions in a partition of unity are nonnegative.) Set $\delta = \sum \rho_i |z_i| + \sigma$. Show that $\delta|_D = 0$ and $\delta|_{X \setminus U} = 1$.
(3) Let V be an open set meeting D and w is a holomorphic function on V vanishing on D. Show that $|w|/\delta$ extends to a continuous nonnegative function on V. Show that, if w vanishes to first order on D and nowhere else, then $|w|/\delta$ is strictly positive.

Now we start working with line bundles. If the partitions of unity were painful to you, you might want to start here:

(4) Define a map of sheaves $\mathcal{O}(-D) \to \mathrm{LC}_{\mathbb{R}}$ by $N : w \mapsto |w|/\delta$. Show that, if $w_1(x) = w_2(x)$ for some point $x \in D$ and some sections w_1 and w_2 of $\mathcal{O}(-D)$ defined near x , then $N(w_1)(x) = N(w_2)(x)$. So N is a norm on the line bundle $\mathcal{O}(-D)$.
(5) For this norm, what is ω? Recall that, if w is any holomorphic section, then $\frac{1}{2\pi i}\partial\overline{\partial} \log N(w)^2$.
(6) In particular, check that ω is supported on U.

[1]I am making X compact for simplicity. In fact, this works for any complex manifold – you just need to be careful to use locally finite covers instead of finite ones.