

PROBLEM SET 8 – DUE WEDNESDAY, NOVEMBER 14

See the course website for policy on collaboration.

1. Let A be a commutative ring and \mathfrak{m} a maximal ideal. We have previously proved Nakayama's lemma in the form: Let M be a finitely generated A -module. If $M/\mathfrak{m}M = 0$, then there is some $f \equiv 1 \pmod{\mathfrak{m}}$ such that $f^{-1}M = 0$. Prove the following variants:
 - (a) Let M_1 and M_2 be finitely generated A -modules and $M_1 \rightarrow M_2$ a map of A -modules. If the induced map $M_1/\mathfrak{m}M_1 \rightarrow M_2/\mathfrak{m}M_2$ is surjective, then there is some $f \equiv 1 \pmod{\mathfrak{m}}$ such that $f^{-1}M_1 \rightarrow f^{-1}M_2$ is surjective.
 - (b) Let M be a finitely generated A -module and let $v_1, v_2, \dots, v_r \in M$ and let \bar{v}_j be the class of v_j in $M/\mathfrak{m}M$. Suppose that $\bar{v}_1, \dots, \bar{v}_r$ span $M/\mathfrak{m}M$ as an A/\mathfrak{m} vector space. Show that there is some $f \equiv 1 \pmod{\mathfrak{m}}$ such that v_1, \dots, v_r generate $f^{-1}M$ as an $f^{-1}A$ module.

2. This question takes place on \mathbb{P}^1 with homogenous coordinates $[z_1 : z_2]$. Let $x = z_1/z_2$. Show that the space of globally regular vector fields on \mathbb{P}^1 is three dimensional, spanned by the vector fields whose values on the open set $\{z_2 \neq 0\}$ are $\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}$.

3. In this question, take $\text{char}(k) \neq 2$. Let $a(x) = a_{2g+1}x^{2g+1} + a_{2g}x^{2g} + \dots + a_2x^2 + a_1x$ be a squarefree polynomial in x with $a_1a_{2g+1} \neq 0$. (Yes, the a_0 is meant to be missing.) Let X be the affine curve $y^2 = a(x)$ in \mathbb{A}^2 , and let A be its coordinate ring.
 - (a) Show that X is smooth.
 - (b) Show that there is a regular 1-form ω_1 on X with $\omega = \frac{dx}{2y} = \frac{dy}{a'(x)}$ wherever the second and third expression are defined. Show that Ω_A^1 is the free A -module with basis ω .
We now restrict to the case $g = 1$. (Higher g will return on later problem sets).
 - (c) Let E be the curve $z_2^2z_3 = a_3z_1^3 + a_2z_1^2z_3 + a_1z_1z_3^2$. Show that $E \cap \{z_3 \neq 0\}$ is isomorphic to X . Show that ω extends to a regular 1-form on all of E .

4. Let $X = Z(x_1x_2) \subset \mathbb{A}^2$. Let A be the coordinate ring, $k[x_1, x_2]/\langle x_1x_2 \rangle$.
 - (a) Show that the 1-form x_1dx_2 vanishes everywhere on TX .
 - (b) Show that there is a well-defined A -linear map $\Omega_A^1 \rightarrow k[x_1, x_2]/\langle x_1^2, x_2^2 \rangle$ sending $dx_1 \mapsto x_1$ and $dx_2 \mapsto -x_2$. Deduce that x_1dx_2 is not zero in Ω_A^1 .

5. This problem introduces the Grassmannian, an object which I sadly did not have the chance to get to in class. Let $0 < d < n$. The Grassmannian $G(d, n)$ will be a variety whose points are in natural bijection with the set of d -planes in k^n . Let $U \subset \mathbb{A}^{dn}$ be the set of $n \times d$ matrices of rank d . Let $\phi : U \rightarrow \mathbb{P}(\bigwedge^d k^n)$ be the map which sends the matrix with columns v_1, v_2, \dots, v_d to $[v_1 \wedge v_2 \cdots \wedge v_d]$.
 - (a) Explain why Problem 7 on Problem Set 6, together with Problem 3 on Problem Set 7, show that the image of ϕ is closed in $\mathbb{P}(\bigwedge^d k^n)$. Calling that image $G(d, n)$, explain why $G(d, n)$ is in bijection with the set of d -planes in n -space.
 - (b) Let e_1, \dots, e_n be the standard basis of k^n , so $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} : 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ is a basis of $\bigwedge^d k^n$. Let $V \subset \mathbb{P}(\bigwedge^d k^n)$ be the open set where the coefficient of $e_1 \wedge e_2 \wedge \cdots \wedge e_d$ is nonzero. Show that each point of $G(d, n) \cap V$ is of the form $\phi\left(\begin{smallmatrix} \text{Id}_d \\ M \end{smallmatrix}\right)$ for a unique $(n-d) \times d$ matrix M .
 - (c) Show that $G(d, n) \cap V \cong \mathbb{A}^{d(n-d)}$.