## Problem Set 2 – due September 21

See the course website for policy on collaboration.

**Notation** Throughout this problem set, $k$ denotes an algebraically closed field.

1. Let $A$ be an $n \times n$ matrix with entries in $k$. Let $R$ be the commutative ring $k[A]$. What is the relationship between the eigenvalues of $A$ (also called the ***spectrum*** of $A$) and $\operatorname{MaxSpec} R$?
2. Prove the Zariski topology on $k^2$ is **not** the product topology of the Zariski topology on $k$ with itself.
3. Let $R$ be a finitely generated $k$-algebra. Choose generators $x_1, x_2, \ldots, x_n$ and write $R = k[x_1, \ldots, x_n]/I$. So we have a natural bijection $\operatorname{MaxSpec}(R) \longleftrightarrow Z(I)$. Show that $Y \subseteq X$ is closed in the Zariski topology if and only if there is an ideal $J \subseteq R$ such that $Y$ corresponds to the set of maximal ideals containing $J$.
4. Describe the images of the following maps. Are they open? Closed?
   (a) Map $k^2$ to $k^2$ by $(x, y) \mapsto (x, xy)$.
   (b) Let $SL_2 = \{ \left( \begin{smallmatrix} w & x \\ y & z \end{smallmatrix} \right) : \ wz - xy = 1 \}$. Map $SL_2$ to $k^2$ by $\left( \begin{smallmatrix} w & x \\ y & z \end{smallmatrix} \right) \mapsto (w, x)$.

**The remainder of this problem set reminds us of the basic properties of Noetherian rings:**

Let $R$ be a commutative ring. We define the following properties of $R$, which we will then show are all equivalent. If $R$ has any (and hence all) of these properties, we define $R$ to be ***noetherian***.

$1(a)$ For any chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in $R$, we have $I_r = I_{r+1}$ for all sufficiently large $r$.

$1(b)$ Every ideal $I$ of $R$ is finitely generated.

$2(a)$ For any $n \geq 0$ and any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of $R^n$, we have $M_r = M_{r+1}$ for all sufficiently large $r$.

$2(b)$ For any $n \geq 0$, every submodule $M$ of $R^n$ is finitely generated.

$3(a)$ For any finitely generated $R$-module $S$ and any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of $S$, we have $M_r = M_{r+1}$ for all sufficiently large $r$.

$3(b)$ For any finitely generated $R$-module $S$, every submodule $M$ of $S$ is finitely generated.

5. For your favorite choice of $\# \in \{1, 2, 3\}$, show that $\#(a)$ and $\#(b)$ are equivalent.
6. For your favorite choice of $* \in \{a, b\}$, show that:
   (a) $1(*) \implies 2(*)$.
   (b) $2(*) \implies 3(*)$.
   (c) $3(*) \implies 1(*)$.

   We remark that $1(b)$ is obvious for fields, but $2(b)$ is the first significant theorem in a linear algebra course, so you should expect to have to do some work.
7. Show that a quotient of a noetherian ring is noetherian. Hint: The $3(*)$ properties are your friend.
8. We will now prove the **Hilbert basis theorem**: If $A$ is noetherian, then $A[t]$ is noetherian. Hence, by induction on $n$, $k[t_1, t_2, \ldots, t_n]$ is noetherian. Applying the previous problem, this shows any finitely generated $k$-algebra is noetherian.

   Let $I$ be an ideal of $A[t]$. We will be proving Property 1(b), that $I$ is finitely generated. Define $I_d$ to be the set of $g \in A$ such that there is an element of $I$ of the form $gt^d + f_{d-1}t^{d-1} + \cdots f_1 t + f_0$.
   (a) Show that $I_d$ is an ideal of $A$.
   (b) Show that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.
   Using property $1(a)$ of $A$, there there is some ideal $I_\infty$ of $A$ so that $I_r = I_{r+1} = \cdots = I_\infty$. Using property $1(b)$ of $A$, take a finite list of generators $g_1, g_2, \ldots, g_k$ of $I_\infty$. For each $g_i$, choose $f_i \in I$ of the form $g_i t^r +$ lower order terms.
   (c) Show that $I \cap A \cdot \{1, t, t^2, \ldots, t^{r-1}\}$ is finitely generated as an $A$-module.
   Let $h_1, h_2, \ldots, h_\ell$ be a list of generators for $I \cap A \cdot \{1, t, t^2, \ldots, t^{r-1}\}$.
   (d) Show that $f_1, f_2, \ldots, f_k, h_1, h_2, \ldots, h_\ell$ generate $I$ as an $A[t]$ module.