MATH 594 PROBLEM SET 5: DUE FEBRUARY 15

See the course website for homework policy.

Problem 1 Let G be a simple group of order 168. How many elements of order 7 does G contain? (This problem does not relate to representation theory; it's just a chance to think a bit more about groups.)

Problem 2 Let G be a finite group and X a finite set with an action of G. Let $\mathbb{C}X$ be the vectors space of functions $X \to \mathbb{C}$, with the obvious action of G. Let χ be the character of $\mathbb{C}X$.

(a) Give a simple description of $\chi(g)$.

(b) Show by a direct argument $\dim(\mathbb{C}X)^G$ is the number of orbits of G acting on X.

(c) Compare (b) with the formula $\dim V^G = \frac{1}{\#(G)} \sum_{g \in G} \chi(g)$ proved in class. What result have you reproved?

(d) Show that $\frac{1}{|G|} \sum_{g \in G} \chi(g)^2$ is the number of orbits of G acting on $X \times X$.

(e) Let G act transitively on X, and let $U \subset \mathbb{C}X$ be the space of functions $f : X \to \mathbb{C}$ with $\sum_{x \in X} f(x) = 0$. Show that U is irreducible if and only if G acts transitively on the set of ordered pairs $\{(x, y) \in X^2,\ x \neq y\}$.

Problem 3 Let G be a group, k a field and V and W finite dimensional representations of G over k. Define $\mathrm{Hom}_G(V, W)$ to be the set of k-linear maps $\phi : V \to W$ such that $\phi(g \cdot v) = g \cdot \phi(v)$ for all $g \in G$ and $v \in V$.

Let K be an extension field of k. Let G act on $V \otimes K$ and $W \otimes K$ by the same matrices by which it acts on V and W. Let $\mathrm{Hom}_G(V \otimes K, W \otimes K)$ be the set of K-linear maps $\psi : V \otimes K \to W \otimes K$ such that $\psi(g \cdot v) = g \cdot \psi(v)$ for all $g \in G$ and $v \in V \otimes K$.

(a) Show that $\dim_k \mathrm{Hom}_G(V, W) = \dim_K \mathrm{Hom}_G(V \otimes K, W \otimes K)$.

(b) (**Harder**) Show that, if k is infinite and $V \otimes K \cong W \otimes K$, then $V \cong W$. Hint: You'll want the following lemma, which you may use without proof: If k is an infinite field, and $f(x_1, x_2, \ldots, x_n)$ is a nonzero polynomial with coefficients in k, then there is a point $(a_1, a_2, \ldots, a_n) \in k^n$ such that $f(a_1, a_2, \ldots, a_n) \neq 0$.

Problem 4 (Optional, will reappear on the next problem set) In this problem we will prove the following result: Let G be a finite group and let V be a faithful finite dimensional representation of G over $\mathbb{C}$. (Faithful means that the map $G \to GL(V)$ is injective.) Let W be any irreducible representation of G. Then there exists an n such that W is a summand of $V^{\otimes n}$.

(a) Let χ_V and χ_W be characters of V and W. Express $\dim \mathrm{Hom}_G(W, V^{\otimes n})$ in terms of χ_V, χ_W and n.

(b) Show that, for all $g \in G$, we have $|\chi_V(g)| \leq \dim V$. (Hint: What are the eigenvalues of g acting on V?)

(c) Suppose that, for all g in G other than 1_G, we have $|\chi_V(g)| < \dim V$. With this hypothesis, show that $\dim \mathrm{Hom}(W, V^{\otimes n}) > 0$ for n sufficiently large.

(d) Show that, if there is a nonzero map $W \to V^{\otimes n}$, then that map is an injection.

(e) Parts (c) and (d) would solve the problem if it were true that $|\chi_V(g)|$ is strictly less than $\dim V$ for $g \neq 1_G$. Unfortunately, this does not always hold. Give an example of a group G and a faithful representation V such that $|\chi_V(g)| = \dim V$ for some g in G not equal to 1_G.

(f) Define V' to be the representation $V \oplus \mathbb{1}$ of G. Show that, for all g in G other than 1_G, we have $|\chi_{V'}(g)| < \dim V'$.

(g) Show that there exists an n such that W is a summand of $V^{\otimes n}$.