MATH 594 PROBLEM SET 4: DUE FEBRUARY 8

See the course website for homework policy.

Problem 1 Let V be a vector space over an algebraically closed ground field. Let A be an abelian group and let $\rho : A \to GL(V)$ be a simple G-representation.

(a) Show that $\rho(a)$ is a scalar multiple of the identity matrix for all $a \in A$.

(b) Show that V is one dimensional.

Problem 2 If one knows the characteristic polynomial of an endomorphism $T : V \to V$ of a finite dimensional vector space V, can one determine the characteristic polynomials of the exterior powers $\bigwedge^m T : \bigwedge^m V \to \bigwedge^m V$ for all $m > 1$? If so, give a proof; if not, give a counterexample.

Problem 3 Let $\rho : G \to GL(V)$ be a representation with character χ.

(a)Let ψ be the character of $\bigwedge^2 V$. Prove the identity:

$$\psi(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right).$$

(b) Find a similar identity expressing the character of $\bigwedge^3 V$ in terms of $\chi(g)$, $\chi(g^2)$ and $\chi(g^3)$.

Problem 4 Let G be a group and let X be a set on which G acts. Let $\mathbb{C}X$ be the $\mathbb{C}$ vector space with basis e_x indexed by $x \in X$. Let G act on $\mathbb{C}X$ by $g \cdot e_x = e_{gx}$.

(a) Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. Find two six element sets X and Y on which G acts, so that $\mathbb{C}X$ and $\mathbb{C}Y$ have the same character, but there is no bijection $X \to Y$ commuting with the G action.

(b) Give an explicit isomorphism between the representations $\mathbb{C}X$ and $\mathbb{C}Y$ of G.

(c) Using your construction from part (a), we get two maps from $(\mathbb{Z}/2)^2$ into S_6. Let H_1 and H_2 be the images of these maps. Let X_1 and X_2 be S_6/H_1 and S_6/H_2. Show that $\mathbb{C}X_1$ and $\mathbb{C}X_2$ have the same character as S_6 representations.

(d) Show that there is no bijection $X_1 \to X_2$ commuting with the S_6 action.

The example in parts (c) and (d) shows that this phenomenon can happen even when the actions are transitive.